

NUCLEONS AS CHIRAL SOLITONS

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In the limit of large number of colors N_c the nucleon consisting of N_c quarks is heavy, and one can treat it semiclassically, like the large- Z Thomas–Fermi atom. The role of the semiclassical field binding the quarks in the nucleon is played by the pion or chiral field; its saddle-point distribution inside the nucleon is called the chiral soliton. The old Skyrme model for this soliton is an over-simplification. One can do far better by exploiting a realistic and theoretically-motivated effective chiral lagrangian presented in this paper. As a result one gets not only the static characteristics of the nucleon in a fair accordance with the experiment (such as masses, magnetic moments and formfactors) but also much more detailed dynamic characteristics like numerous parton distributions. We review the foundations of the Chiral Quark-Soliton Model of the nucleon as well as its recent applications to parton distributions, including the recently introduced ‘skewed’ distributions, and to the nucleon wave function on the light cone.

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1 Introduction

Application of the QCD sum rules¹ to nucleons, pioneered by B.L. Ioffe² has taught us several important lessons. One is that the physics of nucleons is heavily dominated by effects of the Spontaneous Chiral Symmetry Breaking (SCSB). One sees it from the fact that all Ioffe's formulae for nucleon observables, including the nucleon mass itself, is expressed through the SCSB order parameter $\langle\bar{\psi}\psi\rangle$. Therefore, it is hopeless to build a realistic theory of the nucleon without taking into due account the SCSB.

The chiral quark-soliton model of the nucleon we are going to review in this paper is based on two ideas. One is the dominating role the SCSB plays in the dynamics of the nucleon bound state. The second is, in fact, a technical tool, not a matter of principle. The idea is that our world with the number of colors $N_c = 3$ is not qualitatively different from an imaginary world with large number of colors. If one is able to build a nucleon as a bound state of N_c quarks at large N_c (keeping N_c as a free algebraic parameter) and if, furthermore, $1/N_c$ corrections are under control, one can reasonably expect that the picture one gets putting $N_c = 3$ is not too far from reality. We shall see that actually in some cases the leading- N_c predictions are satisfied in nature to a 1% accuracy (!), that is far better than one might expect beforehand from a generic $1/N_c \approx 30\%$ correction.

We start from surveying the background of the model: spontaneous chiral symmetry breaking and the effective chiral lagrangian ($E\chi L$) including the Wess-Zumino term. We formulate a simple but realistic $E\chi L$ which is compatible with the phenomenology and follows from certain theoretical considerations. The properties of this $E\chi L$ is studied in some detail. For that purpose we introduce a technique of gradient expansion of functional determinants, which is widely used in many other physical applications. We then build the semi-classical model of the nucleon, justified at large N_c , and present another technical tool, that of the quantization of soliton rotations. In the context of the nucleon problem it is necessary to describe different spin-isospin states of a nucleon as well as that of the Δ resonance. Finally, we briefly describe recent development in the field, namely, the calculation of various parton distributions inside the nucleon at low virtuality. Parton distributions obtained in the model satisfy all general requirements: they are positive, satisfy appropriate sum rules, etc. In addition, they appear to be in a good accordance with the data (where measured) without any fitting parameters whatsoever.

2 How do we know chiral symmetry is spontaneously broken?

The QCD lagrangian with N_f massless flavors is known to possess a large global symmetry, namely a symmetry under $U(N_f) \times U(N_f)$ independent rotations of left- and right-handed quark fields. This symmetry is called *chiral*. [The word was coined by Lord Kelvin in 1894 to describe molecules not superimposable on its mirror image.] Instead of rotating separately the 2-component Weyl spinors corresponding to left- and right-handed components of quark fields, one can make independent vector and axial $U(N_f)$ rotations of the full 4-component Dirac spinors – the QCD lagrangian is invariant under these transformations too.

Meanwhile, axial transformations mix states with different P-parities. Were that symmetry exact, one would observe parity degeneracy of all states with otherwise the same quantum numbers. In reality the splittings between states with the same quantum numbers but opposite parities are huge. For example, the splitting between the vector ρ and the axial a_1 meson is $(1260 - 770) \simeq 500$ MeV; the splitting between the nucleon and its parity partner is even larger: $(1535 - 940) \simeq 600$ MeV.

The splittings are too large to be explained by the small bare or current quark masses which break the chiral symmetry from the beginning. Indeed, the current masses of light quarks are: $m_u \simeq 4$ MeV, $m_d \simeq 7$ MeV, $m_s \simeq 150$ MeV. The only conclusion one can draw from these numbers is that the chiral symmetry of the QCD lagrangian is broken down *spontaneously*, and very strongly. Consequently, one should have light (pseudo) Goldstone pseudoscalar hadrons – their role is played by pions which indeed are by far the lightest hadrons.

The order parameter associated with chiral symmetry breaking is the so-called *chiral* or *quark condensate*:

$$\langle \bar{\psi}\psi \rangle \simeq -(250 \text{ MeV})^3 \quad (1)$$

at the scale of a few hundred MeV. It should be noted that this quantity is well defined only for massless quarks, otherwise it is somewhat ambiguous. By definition, this is the quark Green function taken at one point; in momentum space it is a closed quark loop. If the quark propagator has only the ‘slash’ term, the trace over the spinor indices understood in this loop gives an identical zero. Therefore, chiral symmetry breaking implies that a massless (or nearly massless) quark develops a non-zero dynamical mass (i.e. a ‘non-slash’ term in the propagator). There are no reasons for this quantity to be a constant independent of the momentum; moreover, we understand that it should anyhow vanish at large momentum. The value of the dynamical mass at small virtuality

can be estimated as one half of the ρ meson mass or one third of the nucleon mass, that is about

$$M(0) \approx 350 \text{ MeV}; \quad (2)$$

this quantity is directly related to chiral symmetry breaking and should emerge together with the condensate (1) in any theoretical description of the SCSB.

3 Effective Chiral Lagrangian (E χ L)

Let us consider QCD in the chiral limit, meaning that we put the light current quark masses to zero. For u, d quarks this is a good approximation to reality: it is known that most physical observables computed in the chiral limit differ from their true values by about 5%, unless the quantity is for some special reasons divergent in the chiral limit. For example, the nucleon mass in the chiral limit is lighter by about 5%: this is the contribution of the σ -term to the nucleon mass. For the s quark this idealization has worse accuracy, about 20%. We would like to keep the number of light flavors N_f a free parameter putting it equal to 3 or 2 depending on whether we incorporate strangeness or not.

Once chiral symmetry is spontaneously broken, the lightest degrees of freedom in QCD are $N_f^2 - 1$ pseudoscalar Goldstone bosons (we shall call them ‘pions’ for short even in the case of $N_f = 3$ where four kaons and the η meson are also Goldstone bosons). Therefore, in the low-momenta region QCD is reduced to a theory of massless interacting Goldstone bosons. The appropriate nonlinear lagrangian describing Goldstone pions is called the effective chiral lagrangian (E χ L). It is convenient to write the pion fields in terms of a unitary $(N_f^2 - 1) \times (N_f^2 - 1)$ matrix,

$$U(x) = \exp i \frac{\lambda^A \pi^A(x)}{F_\pi}, \quad (3)$$

where λ^A are eight Gell-Mann matrices in case we consider $N_f = 3$ flavors and three Pauli matrices for $N_f = 2$. We divide the exponent by a pion decay constant $F_\pi = 94 \text{ MeV}$ in order to use a properly normalized pion field with a standard kinetic energy, see below.

3.1 General properties

In the chiral limit pions are strictly massless, so the E χ L needs to have at least two derivatives of the matrix U , or more. Another general requirement

for building the E χ L is that it should be chiral-invariant, meaning invariance under a global $SU_L(N_f) \times SU_R(N_f)$ rotation,

$$U \rightarrow AUB^\dagger, \quad U^\dagger \rightarrow BU^\dagger A^\dagger, \quad (4)$$

with constant $SU(N_f)$ matrices A, B . It is useful to introduce hermitian matrices

$$L_\mu = iU\partial_\mu U^\dagger = \frac{1}{F_\pi}\partial_\mu\pi^A\lambda^A + \dots \quad (5)$$

transforming under the rotation (4) as $L_\mu \rightarrow AL_\mu A^\dagger$.

A general form of the E χ L can be presented as a series in the number of derivatives of the fields; it is called the *derivative expansion*. The leading term is the two-derivative one; there is only one possibility to write it, compatible with the invariance (4):

$$\begin{aligned} \mathcal{L}^{(2)} &= \frac{F_\pi^2}{4} \text{Tr} L_\mu L_\mu = \frac{F_\pi^2}{4} \text{Tr} \partial_\mu U^\dagger \partial_\mu U \\ &= \frac{1}{2} (\partial_\mu \pi^A)^2 + \frac{1}{3F_\pi^2} f^{ABE} f^{CDE} \partial_\mu \pi^A \pi^B \partial_\mu \pi^C \pi^D + O(\pi^6). \end{aligned} \quad (6)$$

The first term is the usual kinetic energy for pions, while the second (quartic term) describes the p -wave pion scattering. The Noether current associated with the left-hand transformation in (6) by an x -dependent matrix A is

$$J_{L\mu}^A = -\frac{F_\pi^2}{2} \text{Tr} (L_\mu \lambda^A) = -F_\pi \partial_\mu \pi^A + \dots \quad (7)$$

Charged pions decay owing to this current which couples to the W^\pm bosons, hence the normalization factor F_π in the above equations.

In the four-derivative order there are several invariants,

$$\begin{aligned} \mathcal{L}^{(4)} &= L_1 \text{Tr} (L_\mu L_\mu) \text{Tr} (L_\nu L_\nu) + L_2 \text{Tr} (L_\mu L_\nu) \text{Tr} (L_\mu L_\nu) \\ &+ L_3 \text{Tr} (L_\mu L_\mu L_\nu L_\nu) + C_4 \text{Tr} (\partial_\mu L_\mu)^2 \end{aligned} \quad (8)$$

where we introduce the coefficients L_1, \dots in accordance with notations of Gasser and Leutwyler.³ These coefficients have been fitted to the d -wave pion scattering and other low-energy data, and their present-day values are:⁴

$$L_1 \approx 2L_2, \quad L_2 = (1.35 \pm 0.3)10^{-3}, \quad L_3 = (-3.5 \pm 1.1)10^{-3}. \quad (9)$$

The equation $L_1 = 2L_2$ is a prediction of large N_c ; the term with C_4 is usually expelled from the four-derivative family as it affects the physical amplitudes only at higher orders; at $N_f = 2$ the three first terms in Eq. (8) are not independent. For these reasons the four-derivative terms of the E χ L are sometimes presented as

$$\begin{aligned}\mathcal{L}^{(4)} &= (2L_2 + L_3)\text{Tr}(L_\mu L_\mu L_\nu L_\nu) + L_2\text{Tr}(L_\mu L_\nu L_\mu L_\nu) \\ &= (3L_2 + L_3)\text{Tr}(L_\mu^2 L_\nu^2) + \frac{L_2}{2}\text{Tr}[L_\mu L_\nu]^2\end{aligned}\quad (10)$$

where $2L_2 + L_3$ is compatible with zero; the last term is called the Skyrme term.

In the six-derivative order there are rather many invariants but only limited information is known about them from phenomenology.⁵

3.2 Wess–Zumino term

There is another famous term in the four-derivative order, called the Wess–Zumino term.⁶ It has several interesting peculiarities. First, it cannot be written down in a local form as Eqs.(6, 8). One of the forms has been suggested by Eides and one of the authors⁷ as an integral over a parameter:

$$\begin{aligned}\mathcal{L}^{WZ} &= \frac{N_c}{24\pi^2} \int_0^1 ds \epsilon_{\alpha\beta\gamma\delta} \text{Tr} \left(e^{-is\Pi} \partial_\alpha e^{is\Pi} \right) \begin{pmatrix} \end{pmatrix}_\beta \begin{pmatrix} \end{pmatrix}_\gamma \begin{pmatrix} \end{pmatrix}_\delta \\ &= \frac{2N_c}{15\pi^2} \text{Tr} \left(\Pi \partial_\alpha \Pi \partial_\beta \Pi \partial_\gamma \Pi \partial_\delta \Pi \right) + O(\pi^7), \quad \Pi = \pi^A \lambda^A / F_\pi.\end{aligned}\quad (11)$$

This expression is nonzero only starting from $N_f \geq 3$. The leading $O(\pi^5)$ term gives the low-energy amplitude of the process $\bar{K}K \rightarrow \pi\pi\pi$. For two flavors the Wess–Zumino term is zero, at least perturbatively, in the sense that all terms of the expansion in powers of the pion field are zero. However, as shown by Witten,⁹ for certain ‘large’ fields it is nonzero, even for $U \in SU(2)$. Also, the variational derivatives of Eq. (11) are nonzero. For example, one may wish to ‘gauge’ the Wess–Zumino action by replacing the derivatives in Eq. (11) by covariant derivatives of the flavor $SU_L(N_f) \times SU_R(N_f)$ group. This is necessary if one likes to learn how pions couple to the photon and the W, Z bosons. The task of gauging Eq. (11) is not a trivial one⁹ given the nonlocal form of the Wess–Zumino action. The complete and correct result has been given by Dhar, Shankar and Wadia.⁸ The gauged Wess–Zumino action is nonzero for any N_f including $N_f = 2$. It gives, in particular, the low-momenta limit of the

processes $\pi^0 \rightarrow \gamma\gamma$ and $\gamma \rightarrow \pi\pi\pi$. In a more common language these processes are determined by the axial anomaly. Therefore, the Wess–Zumino term in the $E\chi L$ incorporates the axial anomaly, but in a $SU_L(N_f) \times SU_R(N_f)$ invariant way.

Last, let us present the Wess–Zumino term in a 5-dim form suggested by Witten.⁹ The action corresponding to the lagrangian (11) can be written as

$$S^{WZ}[\pi] = -\frac{N_c}{240\pi^2} \int d^5x \epsilon_{\alpha\beta\gamma\delta\epsilon} \text{Tr} \left(L_\alpha L_\beta L_\gamma L_\delta L_\epsilon \right). \quad (12)$$

In this equation it is understood that the pion field U is continued without singularities from the physical 4-dim space-time to a fifth dimension such that the physical space-time serves as a border of the 5-dim ‘disk’. Actually, it can be shown that the integrand in Eq. (12) is a full derivative, so that Eq. (12) does not depend on the concrete way one continues the pion field inside the ‘disk’. The above Eq. (11) can be viewed as a specific continuation procedure, with the parameter s playing the role of the ‘disk’ radius.

If the light current quark masses are nonzero, the pion fields become massive (they are called pseudo-Goldstone fields in this case), and there appear terms in the $E\chi L$ without derivatives,

$$\text{Tr } m(U + U^\dagger), \quad \text{Tr } mUmU^\dagger, \quad \text{Tr } m(U + U^\dagger)L_\mu L_\mu, \quad \text{etc.} \quad (13)$$

where m is the matrix of current quark masses. However, in this paper we shall restrict ourselves to the chiral limit and neglect such terms.

Were QCD exactly solvable, one would be able to derive all terms of the derivative expansion of the $E\chi L$. The dimensionfull coefficients (like F_π , etc.) should be expressible through the only dimensional constant one has in the chiral limit of QCD, namely Λ_{QCD} , this quantity appearing, via the transmutation of dimensions, as a renormalization-invariant combination of the ultra-violet cutoff μ and the gauge coupling given at this cutoff,

$$\begin{aligned} \Lambda_{\text{QCD}} &= \mu \left[\frac{16\pi^2}{bg^2(\mu)} \right]^{\frac{b'}{2b^2}} \exp \left[-\frac{8\pi^2}{bg^2(\mu)} \right] \left[1 + O \left(\frac{1}{g^2(\mu)} \right) \right], \\ b &= \frac{11}{3}N_c - \frac{2}{3}N_f, \quad b' = \frac{34}{3}N_c^2 - \frac{13}{3}N_cN_f + \frac{N_f}{N_c}. \end{aligned} \quad (14)$$

So far QCD is not exactly solved, even in the large N_c limit, therefore, to build a realistic $E\chi L$ one has to rely upon models, physical considerations and comparison with phenomenology.

3.3 Skyrme model

A very simple model is named after Skyrme.¹⁰ It consists in truncating the infinite series of the derivative expansion at the four-derivative level, moreover, leaving only one term in Eq. (8), the ‘Skyrme term’,

$$\mathcal{L}^{\text{Skyrme}} = \frac{F_\pi^2}{4} \text{Tr} L_\mu L_\mu + \frac{1}{32e^2} \text{Tr} [L_\mu L_\nu]^2. \quad (15)$$

Skyrme has suggested this simplified version of the chiral lagrangian to find a static configuration of the pion field minimizing the appropriate energy functional. He conjectured that this static configuration should be associated with the nucleon. More than two decades later Witten⁹ has revitalized the original Skyrme’s idea showing that, if supplemented by the Wess–Zumino term (12), a static local minimum of the $E\chi\text{L}$ can be, indeed, associated with a nucleon in the large- N_c limit of QCD. The model of a nucleon based on the local minimum of Eq. (15) has been presented by Adkins, Nappi and Witten¹¹ and called the *skyrmion*.

Simple as it is, the model cannot be fully realistic because we know that there are other terms with four derivatives, numerically of the same magnitude as the commutator term in (15), not to mention higher-derivative terms. Adding the Wess–Zumino term simply ‘by hands’ looks somewhat *ad hoc*, as is the choice of the simplified $E\chi\text{L}$.

3.4 Low-momenta limit of QCD

It has been first noticed in ref. ⁷ that the Wess–Zumino term can be obtained as a functional integral over the quark fields. We shall show it later in section 4 and now let us consider the interaction of quarks with the chiral field. As explained in section 2, an inevitable consequence of SCSB is the appearance of a dynamical or constituent quark mass M , a ‘non-slash’ term in its propagator. A free lagrangian,

$$\mathcal{L}^{\text{free}} = \bar{\psi}(i\rlap{\not{\partial}} - M)\psi, \quad (16)$$

is, however, not invariant under axial rotations,

$$\psi \rightarrow \exp(i\alpha^A \lambda^A \gamma_5) \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \exp(i\alpha^A \lambda^A \gamma_5), \quad (17)$$

because of the mass term. To construct a lagrangian which is invariant under axial rotations Goldstone pion fields need to be involved. The simplest lagrangian invariant under (17) is

$$\mathcal{L} = \bar{\psi}(i\rlap{\not{\partial}} - MU^{\gamma_5})\psi, \quad (18)$$

$$U^{\gamma_5} \equiv \exp i \frac{\pi^A \lambda^A \gamma_5}{F_\pi} = U \frac{1 + \gamma_5}{2} + U^\dagger \frac{1 - \gamma_5}{2}, \quad (19)$$

since the rotation of the quark fields here can be compensated by renaming of pion fields, $U \rightarrow AUA$ where $A = \exp(i\alpha^A \lambda^A)$.

The dynamical quark mass M is not a constant: we know that at large space-like quark virtualities $M(k)$ has to vanish because of asymptotic freedom, to mention one reason. Therefore, the interaction of quarks with chiral fields is necessarily momentum-dependent, i.e. non-local. To avoid difficult questions of what is going on at time-like virtualities we prefer to formulate the theory first in the space-like region, i.e. in Euclidean space. In passing from Minkowski to Euclidean space we make the substitutions:

$$\begin{aligned} i \int d^4 x_M &= \int d^4 x_E, & i\bar{\psi}_M &= \psi_E^\dagger, & p_M^2 &= -p_E^2, \\ \gamma_{0M} &= \gamma_{4E}, & \gamma_{iM} &= i\gamma_{iE}, & \gamma_{5M} &= \gamma_{5E}, \\ \mathcal{L}_M &= \bar{\psi}(i\not{\partial} - MU^{\gamma_5})\psi \rightarrow \mathcal{L}_E = \psi^\dagger(i\not{\partial} + iMU^{\gamma_5})\psi. \end{aligned} \quad (20)$$

We write the partition function to which the full QCD partition function is reduced at low momenta as a functional integral over both quark and pion fields:^{12,13}

$$\begin{aligned} \mathcal{Z} &= \int D\pi^A \int D\psi^\dagger D\psi \exp \int d^4 x [\psi^\dagger(x) i\not{\partial} \psi(x) \\ &+ i \int \frac{d^4 k_1 d^4 k_2}{(2\pi)^8} e^{i(k_1 - k_2, x)} \psi^\dagger(k_1) \sqrt{M(k_1)} U^{\gamma_5}(x) \sqrt{M(k_2)} \psi(k_2)]. \end{aligned} \quad (21)$$

Eq. (21) shows quarks interacting with chiral fields $U(x)$, with formfactor functions equal to the square root of the dynamical quark mass attributed to each vertex where $U(x)$ applies. U^{γ_5} is a $N_f \times N_f$ matrix in flavor and a 4×4 matrix in Dirac indices; it is a unity $N_c \times N_c$ matrix in color.

3.5 Digression: chiral symmetry breaking by instantons

The partition function (21) has been actually derived in Ref.(12,13) from considering light quarks in the QCD instanton vacuum.¹⁴ Instantons – large topological fluctuations of the gluon field – provide a neat and phenomenologically successful microscopic mechanism of SCSB.¹⁵ Because of the famous 't Hooft zero modes¹⁶ the random instanton ensemble can be viewed as a collection of 'impurities' each binding a quark at exactly zero 'energy'. Owing to the quantum-mechanical overlap of quark wave functions quarks can 'hop' from one instanton to another, and get delocalized. The density of quarks at zero

‘energy’ becomes finite; this density is exactly the chiral condensate $\langle\bar{\psi}\psi\rangle$ via the Banks–Casher formula.¹⁷ In more physical terms, when ‘hopping’ from instanton to anti-instanton and so forth, quarks change their helicity or chirality; delocalization means that an infinite number of such flippings are involved, and that generates the ‘non-slash’ term in the quark propagator, i.e. the dynamical mass $M(k)$. This quantity is directly related to the Fourier transform of the ‘t Hooft zero mode in the field of an instanton. One can evaluate basic quantities related to the SCSB, namely $\langle\bar{\psi}\psi\rangle$, $M(k)$ and F_π through the characteristics of the instanton ensemble, that is their density and average size, these quantities, in their turn, being related to Λ_{QCD} (14) from a variational calculation.^{14,18}

Furthermore, instantons induce (non-local) quark interactions; their bosonization leads, at low momenta, to the above partition function (21),¹³ for a review see Ref.(19). However, Eq. (21) is probably of a more general nature. For example, a similar partition function (but without formfactors) follows from the Nambu–Jona-Lasinio model.²⁰ However, contrary to the NJL model, instanton-induced interaction preserve all the symmetries of the original QCD: they are $SU_L(N_f)\times SU_R(N_f)$ invariant (in case of $N_c = N_f = 2$ they are $SU(4)$ invariant) but break explicitly the axial $U_A(1)$ invariance, which is needed to solve the $U(1)$ problem.^{13,21}

3.6 Local field theory at low momenta

However realistic is the instanton mechanism of SCSB, actually the low-energy partition function (21) is of a very general nature and in principle one needs not refer to instantons to use it. To make it even more general one can add to Eq. (21) a $Z(k)$ factor in the kinetic-energy term for quarks and an additional formfactor function for pions in the interaction term. However, such modifications can be removed by a redefinition of the ψ and U fields.

The formfactor functions $\sqrt{M(k)}$ for each quark line attached to the chiral vertex automatically cut off momenta at some characteristic scale. In the instanton derivation of Eq. (21) this scale is the inverse average size of instantons, $1/\bar{\rho} \approx 600$ MeV. In the range of quark momenta $k \ll 1/\bar{\rho}$ one can neglect this non-locality, and the partition function (21) is simplified to a local field theory:

$$\mathcal{Z} = \int D\pi^A \int D\psi^\dagger D\psi \exp \int d^4x \psi^\dagger(x) [i\cancel{\partial} + iMU^{\gamma_5}(x)] \psi(x). \quad (22)$$

One should remember, however, to cut the quark loop integrals at $k \approx 1/\bar{\rho} \approx 600$ MeV. Notice that there is no kinetic energy term for pions: it appears after one integrates over the quark loop, see below. Summation over color is assumed in the exponent of Eq. (22).

Eq. (22) defines a simple local field theory though it is still a highly non-trivial one. Its main properties will be established in the next section. We shall see that it reproduces the known properties of the E χ L discussed above. After that we shall move to building the nucleon from Eq. (22) ^a.

4 Properties of the effective chiral lagrangian

Properties of effective theories of quarks interacting with various meson fields have been studied by several authors in the 80's, most notably by Volkov and Ebert²³ and Dhar, Shankar and Wadia.⁸ The fact that integrating over quarks one gets, in particular, the Wess–Zumino term has been first established by Diakonov and Eides,⁷ see also below.

Integrating over the quark fields in Eq. (22) one gets an effective chiral lagrangian (E χ L):

$$S_{\text{eff}}[\pi] = -N_c \ln \det (i\cancel{D} + iMU^{\gamma_5}). \quad (26)$$

The Dirac operator in Eq. (26) is not hermitean:

$$D = i\cancel{D} + iMU^{\gamma_5}, \quad D^\dagger = i\cancel{D} - iMU^{\gamma_5^\dagger}, \quad (27)$$

therefore the effective action has an imaginary part. The real part can be defined as

$$\text{Re } S_{\text{eff}}[\pi] = -\frac{N_c}{2} \ln \det \left(\frac{D^\dagger D}{D_0^\dagger D_0} \right),$$

$$D^\dagger D = -\partial^2 + M^2 - M(\cancel{D}U^{\gamma_5}), \quad D_0^\dagger D_0 = -\partial^2 + M^2. \quad (28)$$

^aWe have been asked about the relation of this low-energy theory with that suggested by Manohar and Georgi²². One can redefine the quark fields

$$\psi \rightarrow \psi' = \exp(i\pi^A \lambda^A \gamma_5/2) \psi, \quad \psi^\dagger \rightarrow \psi'^\dagger = \psi^\dagger \exp(i\pi^A \lambda^A \gamma_5/2), \quad (23)$$

and rewrite the lagrangian in (22) as

$$\mathcal{L} = \psi'^\dagger (i\cancel{D} + V + A\gamma_5 + iM) \psi' \quad (24)$$

with

$$V_\mu = \frac{i}{2}(\xi \partial_\mu \xi^\dagger + \xi^\dagger \partial_\mu \xi), \quad A_\mu = \frac{i}{2}(\xi \partial_\mu \xi^\dagger - \xi^\dagger \partial_\mu \xi), \quad \xi = \exp(i\pi^A \lambda^A/2) = U^{1/2}, \quad (25)$$

which resembles closely the effective lagrangian of Manohar and Georgi (the effective chiral lagrangian in a similar form has been independently suggested in Ref.(7)).

The crucial difference is that Manohar and Georgi have added an explicit kinetic energy term $F_\pi^2 \text{Tr}(\partial_\mu U^\dagger \partial_\mu U)/4$ on top of Eq. (24). This seems to be unnecessary as the kinetic energy term arises from quark loops, see the next section.

In the next two subsections we establish the properties of the real and imaginary parts of the E χ L separately, following Ref.(24).

4.1 Derivative expansion and interpolation formula

There is no general expression for the functional (26) for arbitrary pion fields. For certain pion fields the functional determinant (26) can be estimated numerically, see section 5. However, one can make a systematic expansion of the E χ L in increasing powers of the derivatives of the pion field, ∂U . It is called long wave-length or derivative expansion. In fact, one can do it better and expand the real part of the E χ L in powers of

$$\frac{pM}{p^2 + M^2} (U - 1) \quad (29)$$

where p is the characteristic momentum of the pion field. This quantity becomes small in three limiting cases: (i) small pion fields, $\pi^A(x) \ll 1$, with arbitrary momenta, (ii) arbitrary pion fields but with small gradients or momenta, $p \ll M$, (iii) arbitrary pion fields and large momenta, $p \gg M$. We see thus that expanding the E χ L in this parameter one gets accurate results in three corners of the Hilbert space of pion fields. For that reason we call it *interpolation* formula.²⁴ Our experience is that its numerical accuracy is quite good for more or less arbitrary pion fields, even if one uses only the first term of the expansion in (29), see below.

The starting point for both expansions is the following formal manipulation with the real part of the E χ L (28). The first move is to use the well-known formula, $\ln \det[\text{operator}] = \text{Sp} \ln[\text{operator}]$, where Sp denotes a functional trace. One can write:

$$\begin{aligned} \text{Re } S_{\text{eff}}[\pi] &= -\frac{N_c}{2} \ln \det [1 - (-\partial^2 + M^2)^{-1} M(\not{\partial} U^{\gamma_5})] \\ &= -\frac{N_c}{2} \text{Sp} \ln [1 - (-\partial^2 + M^2)^{-1} M(\not{\partial} U^{\gamma_5})] \\ &= -\frac{N_c}{2} \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \text{Tr} \ln [1 - (-\partial^2 + M^2)^{-1} M(\not{\partial} U^{\gamma_5})] e^{ik \cdot x} \\ &= -\frac{N_c}{2} \int d^4x \int \frac{d^4k}{(2\pi)^4} \text{Tr} \ln [1 - (k^2 + M^2 - 2ik \cdot \partial - \partial^2)^{-1} M(\not{\partial} U^{\gamma_5})]. \quad (30) \end{aligned}$$

In going from the second to the third line we have written down explicitly what does the functional trace Sp mean: take matrix elements of the operator

involved in a complete basis (here: plane waves, $\exp(ik \cdot x)$), sum over all states (here: integrate over $d^4k/(2\pi)^4$) and take the trace in x . ‘Tr’ stands for taking not a functional but a usual matrix trace, in our case both in flavor and Dirac bispinor indices. In going from the third to the last line we have dragged the factor $\exp(ik \cdot x)$ through the operator, thus shifting all differential operators $\partial \rightarrow \partial + ik$. We have put a unity at the end of the equation to stress that the operator is acting on unity, in particular, it does not differentiate it. Above is a standard procedure for dealing with functional determinants.²⁵

The last line in Eq. (30) can be now expanded in powers of the derivatives of the pion field: it arises from expanding (30) in powers of $\not{\partial}U^{\gamma_5}$ and of $2ik \cdot \partial + \partial^2$. The first non-zero term has two derivatives,

$$\begin{aligned} \text{Re } S_{\text{eff}}^{(2)}[\pi] &= \frac{N_c}{4} \int d^4x \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left(\frac{M \not{\partial} U^{\gamma_5}}{k^2 + M^2} \right)^2 \\ &= \frac{1}{4} \int d^4x \text{Tr} (\partial_\mu U^\dagger \partial_\mu U) \cdot 4N_c \int \frac{d^4k}{(2\pi)^4} \frac{M^2}{(k^2 + M^2)^2}. \end{aligned} \quad (31)$$

It is the kinetic energy term for the pion field or, better to say, the Weinberg chiral lagrangian: actually it contains all powers of the pion field if one substitutes $U(x) = \exp(i\pi^A(x)\lambda^A/F_\pi)$, see Eq. (6). The proportionality coefficient (the last factor in Eq. (31)) is called F_π^2 , experimentally, $F_\pi \approx 94 \text{ MeV}$. The last factor in Eq. (31) is logarithmically divergent; to make it meaningful we have to recall that we have actually simplified the theory as given by Eq. (21) when writing it in the local form (22). Actually, the dynamical quark mass M is momentum-dependent; it cuts the logarithmically divergent integral at $k \approx 1/\bar{\rho}$. Using the numerical values of $\bar{\rho} \approx 600 \text{ MeV}$ and $M \approx 350 \text{ MeV}$ according to what follows from instantons¹² we find

$$F_\pi^2 = 4N_c \int \frac{d^4k}{(2\pi)^4} \frac{M^2}{(k^2 + M^2)^2} \approx \frac{N_c}{2\pi^2} M^2 \ln \frac{1}{M\bar{\rho}} \approx (100 \text{ MeV})^2 \quad (32)$$

being not a bad approximation to the experimental value of F_π . One can at this point adjust the ultra-violet cutoff $1/\bar{\rho}$ to fit exactly the experimental value of F_π . Actually, the two-derivative term is the only divergent quantity in the E χ L: higher derivative terms are all finite.

A more standard way to present the two-derivative term is by using hermitean $N_f \times N_f$ matrices $L_\mu = iU\partial_\mu U^\dagger$. One can rewrite Eq. (31) as

$$\text{Re } S_{\text{eff}}^{(2)}[\pi] = \frac{F_\pi^2}{4} \int d^4x \text{Tr} L_\mu L_\mu, \quad L_\mu = iU\partial_\mu U^\dagger. \quad (33)$$

The next, four-derivative term in the expansion of $\text{Re}S_{\text{eff}}$ is (note that the metric is Euclidean)

$$\text{Re} S_{\text{eff}}^{(4)}[\pi] = -\frac{N_c}{192\pi^2} \int d^4x \left[2 \text{Tr} (\partial_\mu L_\mu)^2 + \text{Tr} L_\mu L_\nu L_\mu L_\nu \right]. \quad (34)$$

These terms describe, in particular, the d -wave $\pi\pi$ scattering lengths, and other observables. They can be compared with the appropriate terms of the general Gasser–Leutwyler expansion, see Eq. (10). In notations of that equation we obtain

$$\begin{aligned} L_2 &= \frac{N_c}{192\pi^2} = 1.58 \cdot 10^{-3}, & vs. & \quad (1.35 \pm 0.3) \cdot 10^{-3} \text{ (exper.)} \\ 2L_2 + L_3 &= 0, & & \quad \text{(experimentally compatible with 0),} \end{aligned}$$

being compatible with the experimental data. The next-to-next-to-leading six derivative terms following from Eq. (30) have been computed in,^{26,27} however, a detailed comparison with phenomenology is still lacking here.

We would like to mention an interesting paper²⁸ where the derivative expansion of the E χ L has been obtained from the Lovelace–Shapiro dual resonance model. For reasons not fully appreciated this model for the $\pi\pi$ scattering gives coefficients in front of the 4-derivative terms numerically close to those following from Eq. (34), but there is a discrepancy at the 6-derivative level.

We now turn to the interpolation formula promised in the beginning of this subsection. One can start from the last line in Eq. (30) and expand it in powers of $M(\not{\partial}U^{\gamma_5})$. It is clear that the actual expansion parameter will be (29). In the first non-zero order we get²⁴

$$\begin{aligned} \text{Re} S_{\text{eff}}^{\text{interpol}}[\pi] &= \frac{N_c}{4} \int d^4x \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[\frac{1}{(k + i\partial)^2 + M^2} M(\not{\partial}U^{\gamma_5}) \right. \\ &\quad \times \left. \frac{1}{(k + i\partial)^2 + M^2} M(\not{\partial}U^{\gamma_5}) \right]. \end{aligned} \quad (35)$$

It will be convenient now to pass to the Fourier transform of the $U(x)$ field understood as a matrix,

$$U(p) = \int d^4x e^{ip \cdot x} [U(x) - 1]. \quad (36)$$

The partial derivatives appearing in Eq. (35) act on the exponents of the Fourier transforms of U, U^\dagger and become corresponding momenta. As a result

we get

$$\begin{aligned} \text{Re } S_{\text{eff}}^{\text{interpol}}[\pi] &= \frac{1}{4} \int \frac{d^4 p}{(2\pi)^4} p^2 \text{Tr} [U^\dagger(p) U(p)] \\ &\times 4N_c \int \frac{d^4 k}{(2\pi)^4} \frac{M^2}{[(k - \frac{p}{2})^2 + M^2] [(k + \frac{p}{2})^2 + M^2]}. \end{aligned} \quad (37)$$

At $p \rightarrow 0$ the last factor becomes F_π^2 (cf. Eq. (32)), and Eq. (37) is nothing but the first term in the derivative expansion, Eq. (31). However, Eq. (37) also describes correctly the functional $S_{\text{eff}}[\pi]$ for rapidly varying pion fields (with momenta $p \gg M$) and for small pion fields of any momenta, when one can anyhow expand Eq. (30) in terms of $\pi^A(x)$ and hence in $U(x) - 1$. The logarithmically divergent loop integral in Eq. (37) should be regularized, as in Eq. (32).

Similarly, one can get the next term in the ‘interpolation’ expansion which will be quartic in $U(p)$, however our experience tells us that already Eq. (37) gives a good approximation to the $\text{E}\chi\text{L}$ for most pion fields.

4.2 The Wess–Zumino term and the baryon number

We now consider the imaginary part of $S_{\text{eff}}[\pi]$. The first non-zero term in the derivative expansion of $\text{Im } S_{\text{eff}}[\pi]$ is ⁸ the Wess–Zumino term. ⁶ It cannot be written as a $d = 4$ integral over a local expression made of the unitary $U(x)$ matrices, however the variation of the Wess–Zumino term is local. For this reason let us consider the variation of $\text{Im } S_{\text{eff}}[\pi]$ in respect to the pion matrix $U(x)$. We have ²⁴

$$\begin{aligned} \delta \text{Im } S_{\text{eff}}[\pi] &= -N_c \delta \text{Im} \ln \det D = \frac{iN_c}{2} \text{Sp} \left(\frac{1}{D} \delta D - \frac{1}{D^\dagger} \delta D^\dagger \right) \\ &= \frac{iN_c}{2} \text{Sp} [(D^\dagger D)^{-1} D^\dagger \delta D - (D D^\dagger)^{-1} D^\dagger \delta D^\dagger]. \end{aligned} \quad (38)$$

Now one can put in explicit expressions for D, D^\dagger from Eqs.(27, 28). The aim of this exercise is to get $\not{\partial}U$ in the denominators so that an expansion in this quantity similar to that of the previous subsection could be used.

Using the Dirac algebra relations (in Euclidean space)

$$\begin{aligned} \{\gamma_\mu, \gamma_\nu\} &= 2\delta_{\mu\nu}, & \gamma_\mu^\dagger &= \gamma_\mu, & \gamma_5 &= \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \gamma_5^\dagger, \\ \{\gamma_5, \gamma_\mu\} &= 0, & \gamma_5^2 &= \gamma_5, & \text{Tr}(\gamma_5 \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta) &= 4\epsilon_{\alpha\beta\gamma\delta}, \end{aligned} \quad (39)$$

one gets after expanding Eq. (38) in powers of $\not{\partial}U^{\gamma_5}$ the first non-zero term

$$\delta \text{Im } S_{\text{eff}}[\pi] = \frac{iN_c}{48\pi^2} \int d^4x \epsilon_{\alpha\beta\gamma\delta} \text{Tr} \left(\partial_\alpha U^\dagger \partial_\beta U \partial_\gamma U^\dagger \partial_\delta U U^\dagger \delta U \right). \quad (40)$$

It can be easily checked that this expression coincides with the variation of the Wess–Zumino term written in the form a $d = 5$ integral⁹

$$\begin{aligned} & \text{Im } S_{\text{eff}}[\pi] \\ &= \frac{iN_c}{240\pi^2} \int d^5x \epsilon_{\alpha\beta\gamma\delta\epsilon} \text{Tr} \left(U^\dagger \partial_\alpha U \right) \left(U^\dagger \partial_\beta U \right) \left(U^\dagger \partial_\gamma U \right) \left(U^\dagger \partial_\delta U \right) \left(U^\dagger \partial_\epsilon U \right) \\ &+ \text{higher derivative terms.} \end{aligned} \quad (41)$$

The integrand in Eq. (41) is a full derivative, however, to write it explicitly one would need some parametrization of the unitary matrix U . The expansion of Eq. (41) starts from the fifth power of $\pi^A(x)$ (it is non-zero only if $N_f \geq 3$, see subsection 3.2). It is important that, similar to $\text{Re } S_{\text{eff}}$, the imaginary part is also an infinite series in the derivatives.

The $E\chi L$ (26) is invariant under vector flavor-singlet transformations. Therefore, there should be a corresponding conserved Noether baryon current, B_μ . This current is associated with the imaginary part of S_{eff} only; since $\text{Im } S_{\text{eff}}$ is an infinite series in the derivatives so is the associated Noether current B_μ . For the Wess–Zumino term (41) the corresponding charge is⁹

$$B = -\frac{1}{24\pi^2} \int d^3\mathbf{x} \epsilon_{ijk} \text{Tr} \left(U^\dagger \partial_i U \right) \left(U^\dagger \partial_j U \right) \left(U^\dagger \partial_k U \right) + \text{higher derivative terms.} \quad (42)$$

The explicitly written term is the winding number of the field $U(x)$. Let us briefly explain this notion.

If $\pi^A(\mathbf{x}) \rightarrow 0$ at spatial infinity so that $U(\mathbf{x}) \rightarrow 1$ in all directions, one can say that the spatial infinity is just one point. Eq. (42) gives then the winding number for the mapping of the three-dimensional sphere S^3 (to which the flat $d = 3$ space is topologically equivalent when ∞ is one point) to the parameter space of the $SU(N_f)$ group. In case $N_f = 2$ the parameter space is also S^3 so that the mapping is $S^3 \mapsto S^3$. The topologically non-equivalent mappings $U(\mathbf{x})$, *i.e.* those which can not be continuously deformed one to another, are classified by their winding number, an integer analytically given by Eq. (42). In case of $N_f > 2$ mathematicians prove that mappings are also classified by integers given by the same Eq. (42).

There exists a prejudice that the baryon number carried by quarks in the external pion field coincides with the winding number of that field. Generally speaking it is not so because of the higher derivative terms omitted in Eq. (42).

Only if the pion field is spatially large and slowly varying so that one can neglect the higher derivative terms in Eq. (42) one can say that the two coincide. Otherwise, for arbitrary pion fields, the baryon number is not related to the winding number: the former may be zero when the latter is unity, and *vice versa*.

To see what is going on here, let us calculate directly the baryon number carried by quarks in an external time-independent pion field $U(\mathbf{x})$.²⁴ The definition of the baryon charge operator in the Minkowski space is

$$\hat{B} = \frac{1}{N_c} \int d^3\mathbf{x} \bar{\psi} \gamma_0 \psi. \quad (43)$$

Passing to Euclidean space (which we prefer to work with since functional integrals are more readily defined in Euclidean) one has to make a substitution $\bar{\psi} \rightarrow -i\psi^\dagger$, $\gamma_0 \rightarrow \gamma_4$, so that

$$\hat{B} = -\frac{i}{N_c} \int d^3\mathbf{x} \psi^\dagger \gamma_4 \psi. \quad (44)$$

The baryon charge in the path integral formulation of the theory given by Eq. (22) is then

$$\begin{aligned} B = \langle \hat{B} \rangle &= -\frac{i}{N_c} \int d^3\mathbf{x} \langle \psi^\dagger \gamma_4 \psi \rangle = -i \int d^3\mathbf{x} \text{Tr} \langle x | \gamma_4 (i\hat{\not{D}} + iMU^{\gamma_5})^{-1} | x \rangle \\ &= -i \int d^3\mathbf{x} \text{Tr} \langle x_4, \mathbf{x} | \frac{1}{i\partial_4 + i\gamma_4\gamma_k\partial_k + iM\gamma_4U^{\gamma_5}} | x_4, \mathbf{x} \rangle \\ &= -i \int d^3\mathbf{x} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \text{Tr} \langle \mathbf{x} | \frac{1}{\omega + iH} | \mathbf{x} \rangle \\ &= \text{Sp } \theta(-H) = \text{number of levels with } \mathbf{E} < 0. \end{aligned} \quad (45)$$

Here

$$H = \gamma_4\gamma_k\partial_k + M\gamma_4U^{\gamma_5} \quad (46)$$

is the Dirac hamiltonian in the external time-independent pion field $U(\mathbf{x})$ and θ is a step function.

Eq. (45) is divergent since it sums up the baryon charge of the whole negative-energy Dirac continuum. This divergence can be avoided by subtracting the baryon charge of the free Dirac sea, *i.e.* with the pion field switched out, $H_0 = \gamma_4\gamma_k\partial_k + M\gamma_4$:

$$B = -i \int d^3\mathbf{x} \int \frac{d\omega}{2\pi} \text{Tr} \langle \mathbf{x} | \frac{1}{\omega + iH} - \frac{1}{\omega + iH_0} | \mathbf{x} \rangle = \text{Sp} [\theta(-H) - \theta(-H_0)]. \quad (47)$$

In performing the integration over ω we have closed the ω integration contour in the upper semiplane. Had we closed it in the lower semiplane we would obtain $-\text{Sp}[\theta(H) - \theta(H_0)]$ which is the same result since $\text{Sp}[\theta(H) + \theta(-H) - \theta(H_0) - \theta(-H_0)] = 0$.

We have thus obtained a most natural result: the baryon charge of quarks in the external pion field is the number of negative-energy levels of the hamiltonian (46) (the number of the levels of the free hamiltonian subtracted).

One can perform the gradient expansion for the baryon number similarly to that of the real part of the $E\chi\text{L}$. To that end let us write

$$B = - \int d^3\mathbf{x} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \text{Tr} \langle x | \frac{H}{\omega^2 + H^2} - \frac{H_0}{\omega^2 + H_0^2} | x \rangle \quad (48)$$

where

$$H^2 = -\partial_k^2 + M^2 - M\gamma_4(\partial_k U^{\gamma_5}), \quad H_0^2 = -\partial_k^2 + M^2. \quad (49)$$

Calculating the matrix element in the plane-wave basis one gets

$$B = - \int d^3\mathbf{x} \int \frac{d\omega}{2\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \text{Tr} \gamma_4 \left[\frac{\gamma \cdot (\partial + i\mathbf{k}) + MU^{\gamma_5}}{\omega^2 - (\partial + i\mathbf{k})^2 + M^2 - M\gamma \cdot (\partial U^{\gamma_5})} - \frac{\gamma \cdot (\partial + i\mathbf{k})}{\omega^2 - (\partial + i\mathbf{k})^2 + M^2} \right] \cdot 1. \quad (50)$$

For slowly varying fields $U(x)$ Eq. (50) can be expanded in powers of ∂U^{γ_5} and ∂ (applied ultimately to U^{γ_5}). Because of the $\text{Tr} \gamma_5 \dots$ the first non-zero contribution arises from expanding the denominator in Eq. (50) to the third power of $\gamma \cdot (\partial U^{\gamma_5})$. Integrals over ω and k are performed explicitly. After some simple algebra one gets

$$B = - \frac{1}{24\pi^2} \int d^3\mathbf{x} \epsilon_{ijk} \text{Tr} (\partial_i U^\dagger \partial_j U \partial_k U^\dagger U) + \text{higher derivative terms} \quad (51)$$

coinciding with Eq. (42) derived from the Noether current corresponding to the Wess-Zumino term (41).²⁹⁻³²

It should be stressed that the baryon number carried by quarks in the background pion field is equal to the topological winding number of the field only if it is a slowly varying one. Mathematically, the reason is that for slowly varying fields one can neglect the higher derivative terms in Eqs.(42, 51). Physically, the reason is the following.²⁴ Imagine we start from a pion field $U(x)$ whose winding number is one but whose spatial size is tending to zero. Such a field has no impact on the spectrum of the Dirac hamiltonian (46): it remains

the same as that of the free hamiltonian, namely it has the upper ($E > M$) and lower ($E < -M$) Dirac continua separated by the mass gap of $2M$. The baryon number is zero.

We now (adiabatically) increase the spatial size of the pion field preserving its winding number equal to unity. Since the winding number is dimensionless this can always be done. At certain critical spatial size the potential well for quarks formed by the external pion field is wide enough so that a bound-state level emerges from the upper continuum. With the increase of the width of the potential well the bound-state level goes down towards the lower Dirac continuum. Asymptotically, as one blows up the spatial size of the pion field (always remaining in the winding number equal unity sector) the bound-state level travels all the way through the mass gap separating the two continua and joins the lower Dirac sea – this is a theorem proven in Ref.(24). At this point one would discover that there is an extra state close to the lower Dirac continuum (as compared to the free, that is no-field case). Therefore, one would say that the baryon number is now unity, – in correspondence to Eqs.(42, 51).

In a general case, however, the baryon number of the quark system is the number of eigenstates of the Dirac hamiltonian (46) one bothers to fill in. The role of the winding number of the background pion field is only to guarantee that, if the spatial size of the field is large enough, the additional bound-state level emerging from the upper continuum is a deep one: asymptotically it goes all the way to the lower continuum.

5 The nucleon

All constituent quark models start from assuming that the nearly massless light quarks of the QCD lagrangian obtain a non-zero dynamical quark mass $M \approx 350$ MeV. This is due to the spontaneous chiral symmetry breaking (SCSB), its microscopic driving force being, to our belief, instantons, see subsection 3.5. Even if one does not believe in instantons as the microscopic mechanism of SCSB one has anyhow to admit that once light quarks get a dynamical or constituent mass $M(k)$ such quarks inevitably have to interact with Goldstone pions, as explained in subsection 3.4.

How strong, numerically, is this interaction? Expanding U^{γ_5} (19) to the first order of the pion field we get from Eq. (18) that the linear coupling of pions to constituent quarks is, numerically, quite large:

$$\begin{aligned}\mathcal{L}^{\text{int}} &= ig_{\pi qq} \pi^A (\bar{\psi} \gamma_5 \lambda^A \psi), \\ g_{\pi qq} &= \frac{M}{F_\pi} \simeq 4.\end{aligned}\tag{52}$$

We would like to emphasize that this is a model-independent consequence of saying that quarks get a constituent mass.

At distances between quarks of the order of 0.5 fm typical for interquark separations inside nucleons, neither the one-gluon exchange nor the supposed linear potential are as large as the chiral forces. Therefore, it is worthwhile to investigate whether the chiral forces alone are able to bind the constituent quarks inside nucleons.

5.1 A note on confinement

The generally accepted confinement scenario is, verbally, very simple. In the pure glue world (without light quarks) it is expected that the Coulomb potential at small separations between static probe quarks is replaced at large separations by a linear potential growing *ad infinitum*. A constant 14 ton force is supposed to be acting between probe quarks even if one is on the Earth and the other one is on the Moon. This is a very radical departure from forces decaying with distance, which have been known to physics so far. If proven correct, it would be one of the most remarkable discoveries in physics of all times.

There have been several proposals of microscopic scenarios which could lead to the linear potential, such as the condensation of magnetic monopoles or a vacuum filled by random $Z(N_c)$ vortices. We are not in a position to discuss them here but would like to note that, despite enormous efforts in the last 25 years, none of them have been worked out mathematically so far, at least at the semi-quantitative level the instanton vacuum has been constructed. The only theoretical model we know of where one gets a linear potential in more than two dimensions is the old example by A.Polyakov³³ of the $d = 2 + 1$ Georgi–Glashow model, but there the non-Abelian color group is broken down to $U(1)$ by the Higgs condensate. Therefore, our belief that the asymptotic linear potential takes place in pure gluodynamics is grounded not on theory but mainly on the results of lattice simulations.

When one looks carefully into how lattice practitioners come to the conclusion about the linear potential one finds several disturbing details. Some of them are discussed in our recent paper³⁴. We mention just one here. The standard way to extract the quark potential on a lattice is by measuring large Wilson loops $W(r, t)$. One side of the loop is called “ r ” (separation), the other one is “ t ” (time). If one wants to measure the potential $V(r)$ at separation r one needs to take $t \gg r$, otherwise the measurement will be contaminated by many excited states. However, such measurements are practically impossible in the interesting range of $r \geq 1$ fm because the average Wilson loop

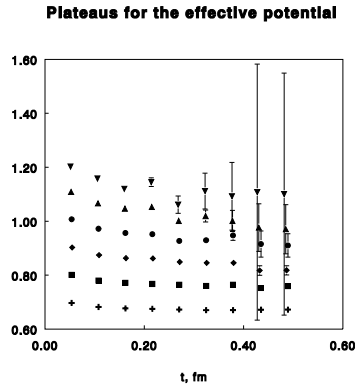


Figure 1: Effective potential $V_{\text{eff}}(r, t) = \ln[W(r, t)/W(r, t + a)]$ as function of t at different values of r from Ref.(35). The data points correspond (from bottom to top) to $r = 0.65, 0.98, 1.30, 1.62, 1.95$ and 2.25 fm. The inverse coupling used $\beta = 2.635$ corresponds to the lattice spacing $a = 0.0541$ fm. Courtesy G.Bali.

becomes so small there that measuring it would require statistics many orders of magnitude exceeding that available with today computers.³⁴

Therefore, this route is abandoned. Instead, Wilson loops are measured in the opposite limit, $t \ll r$. The hope is that the slope $-\partial \ln W(r, t)/\partial t$ reaches its asymptotic ($t \rightarrow \infty$) value $V(r)$ already at small values of t . One tries to assist this early convergence by taking ‘smeared’ links along the r sides of the loop, which are expected to have better overlap with the ground state of the potential. One of the best (if still not the best) measurement of that kind³⁵ is presented in Fig. 1.

One can see from Fig. 1 that the quality of the plateaus at $r \leq 1$ fm is quite good though the data tend to slope down as t increases. At larger r this trend becomes more pronounced, until the error bars explode so that one can hardly get to any conclusions at all. The procedure of extracting the potential is detailed in Ref.(35) but basically it follows from the data in Fig. 1. Thus, the extraction of the linear potential at large r fully relies on the faith in that the ‘plateaus’ of Fig. 1 are not going to slope down as t increases from the region $t \ll r$ where they are measured to $t \gg r$ where they should be measured.

As explained in³⁴ this reliance is potentially dangerous. At $t \ll r$ one can view the Wilson loop’s long side r as ‘time’ and the short side t as ‘separation’; then the linear dependence of $\ln W(r, t)$ on the long side r , i.e. the ‘linear potential’ is built in, whereas it is exactly what is so demanding to

prove. This remark becomes even more disturbing in view of the fact that the linear potential is also observed in two cases where the potential *has* to flatten owing to screening: (i) adjoint sources and (ii) fundamental sources but with light dynamical fermions. In all cases the same procedure of extracting the potentials has been used, namely from measuring Wilson loops at $t \ll r$, and no clear signals of flattening has been observed.

In our mind, the problem of demonstrating the asymptotic linear potential both theoretically and ‘experimentally’ (on the lattice) remains as challenging as it was 25 years ago.

Whatever the outcome of those very important studies, the world we live in is not pure glue: we have light quarks. Light quark-antiquark pairs produced from the vacuum eventually screen any rising potential. In the chiral limit pions are massless, hence the screening starts at the point where the potential exceeds zero energy. In reality the potential cannot exceed $2m_\pi = 280$ MeV which is reached, in the standard (Coulomb + linear potential) parametrization at a rather small 0.3 fm separation. Nucleon is a stationary state; there is an infinite time for any string exceeding that length to decay into pions. Therefore, it seems to be senseless to employ large confining potentials in a realistic model of nucleons.

Why, then, do not quarks ‘get away’ but exist only in bound states? A simple explanation can be provided by the SCSB itself. As a result of SCSB originally massless or nearly massless quarks get a dynamical mass $M \approx 350$ MeV, and pions become the lightest excitations in the spectrum. Therefore, light quarks need not exist as asymptotic states: instead of producing a quark-antiquark pair it is energetically favorable to produce one or several pions. Mathematically, it would correspond to the quark propagator with momentum-dependent mass, having singularities on the second Riemann sheet under the cut starting from the pion threshold.

What about confinement of heavy quarks (c, b, t)? Let us imagine, for the sake of an argument, that the static potential between probe quark sources in a pure glue theory is not rising to infinity but levels off at some $V_\infty = 2\Delta M_\infty$. If ΔM_∞ happens to be larger than approximately the light constituent quark mass M^b heavy quarks would be unstable under decay to D or B mesons. Heavy quarks can be thus confined too. The case is to some extent similar to electrodynamics with charges $Z > 137$: such particles are unstable under a spontaneous production of e^+e^- pairs and therefore cannot exist as asymptotic states.^c

^bThis condition is readily met e.g. in the instanton vacuum.³⁴

^cThe “ $Z > 137$ ” scenario of confinement has been advocated by V.N.Gribov.³⁶

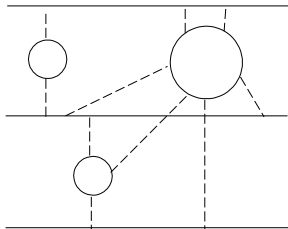


Figure 2: Quarks in a nucleon interacting via pion fields.

5.2 A qualitative picture of the nucleon

Leaving aside those intriguing problems, we concentrate on the lowest state with baryon number one, *i.e.* the nucleon. As mentioned above the interquark separations in the ground-state nucleon are moderate (order of 0.5 fm) and it is worthwhile asking whether the simple $E\chi L$ (22) is capable of explaining the basic properties of the ground-state nucleon. Notice that the expected typical momenta of quarks inside the nucleon are of the order of $M \approx 350$ MeV, that is in the domain of applicability of the low-momentum effective theory (22), see subsection 3.6.

The chiral interactions of constituent quarks in the 3-quark nucleon, as induced by the effective theory (22), are schematically shown in Fig. 2, where quarks are denoted by solid and pions by dashed lines. Notice that, since there is no tree-level kinetic energy for pions in Eq. (22), the pion propagates only through quark loops. Quark loops induce also many-quark interactions indicated in Fig.2 as well. We see that the emerging picture is, unfortunately, rather far from a simple one-pion exchange between the constituent quarks: the non-linear effects in the pion field are not at all suppressed.

At this point one may wonder: isn't the resulting theory as complicated as the original QCD itself? The answer is no, the effective low-energy theory is a simplification as compared to the original quark-gluon theory, because it deals with adequate degrees of freedom. Let us imagine that one would like to describe 'low-energy' properties of solid states, for example superconductivity. Would working with the underlying theory (QED) be helpful? Not too much. We know that the microscopic theory leads, under certain conditions, to the rearrangement of atoms into a lattice, so that translational symmetry is spontaneously broken. As a result the Goldstone bosons appear (here: phonons), and electrons get a dynamical mass different from the input one. The most important forces are due to phonon exchange between electrons: in fact they are driving superconductivity in the BCS theory. Playing with this analogy,

nucleon is like a polaron (a bound state of electrons in the phonon field), rather than a positronium state in the vacuum. After chiral symmetry is broken we deal with a ‘metal’ phase rather than with the vacuum one, and one has to use adequate degrees of freedom to face this new situation.

Instantons play the role of the bridge between the microscopic theory (QCD) and the low-energy theory where one neglects all degrees of freedom except the Goldstone bosons and fermions with the dynamically-generated mass. Instantons do the most difficult part of the job: they explain why atoms in metals are arranged into a lattice, what is the effective mass of the electron and what is the strength of the electron-phonon interactions. However, one can take an agnostic stand and say: I don’t care how 350 MeV is obtained from the microscopic Λ_{QCD} and why do atoms form a lattice in the metals: I just know it happens. To such a person we would advise to take the low-energy theory (22) at face value and proceed to the nucleon.

A considerable technical simplification is achieved in the limit of large N_c . For N_c colors the number of constituent quarks in a baryon is N_c and all quark loop contributions are also proportional to N_c , see section 4. Therefore, at large N_c one can speak about a *classical self-consistent* pion field inside the nucleon: quantum fluctuations about the classical self-consistent field are suppressed as $1/N_c$. The problem of summing up all diagrams of the type shown in Fig. 2 is thus reduced to finding a classical pion field pulling N_c massive quarks together to form a bound state.

5.3 Nucleon mass: a functional of the pion field

Let us imagine that there is a classical time-independent pion field which is strong and spatially wide enough to make a bound-state level of the Dirac hamiltonian (46) for massive quarks, call its energy E_{level} . We fill in this level by N_c quarks in the antisymmetric state in color, thus obtaining a baryon number one state, as compared to the vacuum. The interactions with the background chiral field are color-blind, so one can put N_c quarks on the same level; the fact that one has to put them in an antisymmetric state in color, *i.e.* in a color-singlet state, follows from Fermi statistics.

One has to pay for the creation of this trial pion field, however. Call this energy E_{field} . Since there are no direct terms depending on the pion field in the low-momentum theory (22) the only origin of E_{field} is the fermion determinant (26) which should be calculated for time-independent field $U(\mathbf{x})$. It can be worked out with a slight modification of the equations of section 4. We have:²⁴

$$S_{\text{eff}}[\pi] = -N_c \ln \det \left(\frac{D}{D_0} \right)$$

$$\begin{aligned}
&= -N_c \text{Sp} [\ln(i\partial_t + iH) - \ln(i\partial_t + iH_0)] \\
&= -TN_c \int \frac{d\omega}{2\pi} \text{Sp} [\ln(\omega + iH) - \ln(\omega + iH_0)], \quad (53)
\end{aligned}$$

where H is the Dirac hamiltonian (46) in the stationary pion field, H_0 is the free hamiltonian and T is the (infinite) time of observation. Using an important relation^d

$$\text{Sp}(H - H_0) = 0 \quad (54)$$

(telling us that the sum of all energies, with their signs, of the Dirac hamiltonian (46) is the same as for the free case) one can integrate in Eq. (53) by parts and get

$$\begin{aligned}
S_{\text{eff}}[\pi] &\equiv TE_{\text{field}} = TN_c \int \frac{d\omega}{2\pi} \text{Sp} \left[\frac{\omega}{\omega + iH} - \frac{\omega}{\omega + iH_0} \right] \\
&= TN_c \sum_{E_n^{(0)} < 0} (E_n - E_n^{(0)}). \quad (55)
\end{aligned}$$

Going from the first to the second line we have closed the ω integration contour in the upper semiplane; owing to the trace relation (54) closing it in the lower semiplane would produce the same result.

We see that the price E_{field} one pays for a creation of the time-independent pion field coincides with the aggregate energy of the lower Dirac continuum in that field. The energy of the additional level emerging from the upper continuum, which one has to fill in to get the baryon number one state, E_{level} , should be added to get the total nucleon mass. This simple scheme^{37,24} is depicted in Fig. 3. Naturally, the mass of the nucleon should be counted from the vacuum state corresponding to the filled levels of the free lower Dirac continuum. Therefore, the (divergent) aggregate energy of the free continuum should be subtracted, as in Eq. (55).

We have thus for the nucleon mass:

$$\mathcal{M}_N = \min_{\{\pi^A(\mathbf{x})\}} (N_c E_{\text{level}}[\pi] + E_{\text{field}}[\pi]). \quad (56)$$

Both quantities, E_{level} and E_{field} , are functionals of the trial pion field $\pi^A(\mathbf{x})$. The classical self-consistent pion field is obtained from minimizing the

^dIt follows from taking the matrix trace of the hamiltonian (46). It should be kept in mind, though, that such a naive derivation can be potentially dangerous because of anomalies in infinite sums over levels. However, it can be checked that in this particular case there are no anomalies and the naive derivation is correct.

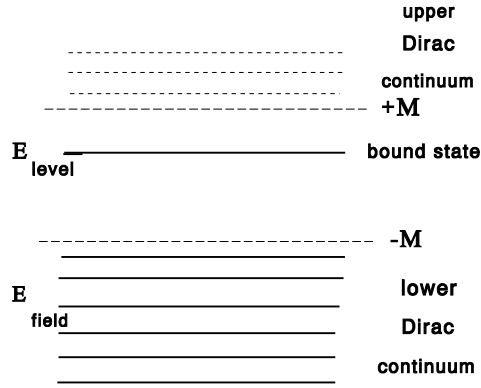


Figure 3: Spectrum of the Dirac hamiltonian in trial pion field. The solid lines show occupied levels.

nucleon mass (56) in $\pi^A(\mathbf{x})$. It is called the *soliton* of the non-linear functional (56), hence the **Chiral Quark-Soliton Model**. An accurate derivation of Eq. (56) from the path-integral representation for the correlation function of Ioffe currents is presented in Ref.(24). It solves the problem of summing up all diagrams of the type shown in Fig. 2 in the large- N_c limit.

The idea that a sigma model with pions coupled directly to constituent quarks, can be used to build the nucleon soliton has been first suggested by Kahana, Ripka and Soni³⁸ and independently by Birse and Banerjee.³⁹ We would call them the authors of the Chiral Quark-Soliton Model. Technically, however, in these references an additional *ad hoc* kinetic energy term for pion fields has been used, leading to a vacuum instability paradox.^{40,41} The present formulation of the model has been given in Ref.(37), together with the discussion of its domain of applicability and its physical contents. A detailed theory based on the path integral approach paving the way for calculating nucleon observables has been presented in Ref.(24).

5.4 Nucleon profile

To find the classical pion field minimizing the nucleon mass (56) one has first of all decide on the symmetry of the pion field. Had the field been a singlet one would take a spherically-symmetric ansatz. However, the pion field has flavor indices $A = 1, \dots, N_f^2 - 1$. At $N_f = 2$ the three components of the pion field can be married with the three space axes. This is called the hedgehog ansatz; it is the minimal generalization of spherical symmetry to incorporate

the $\pi^\pm = (\pi^1 \pm i\pi^2)/\sqrt{2}$ and $\pi^0 = \pi^3$ fields:

$$\pi^A(\mathbf{x})/F_\pi = n^A P(r), \quad n^A = \frac{x^A}{r}, \quad r = |\mathbf{x}|, \quad (57)$$

where $P(r)$ is called the soliton profile.

The choice of the ansatz is not at all innocent: baryons corresponding to different choices would have qualitatively different properties, see the next subsection. The maximally-symmetric ansatz (57) will have definite consequences in applications.

In the $N_f = 3$ case there are 8 components of the ‘pion’ field, and there are several possibilities to marry them to the space axes. The commonly used is the ‘left upper corner’ ansatz:⁴³

$$U(\mathbf{x}) \equiv \exp(i\pi^A(\mathbf{x})\lambda^A) = \begin{pmatrix} \exp[i(\mathbf{n} \cdot \boldsymbol{\tau})P(r)] & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{n} = \frac{\mathbf{x}}{r}. \quad (58)$$

As we shall see in the next subsection, the quantization of rotations for this ansatz leads to the correct spectrum of the lowest baryons.

Another $SU(3)$ ansatz⁴⁴ discussed in the literature is $(f, g, h = 1, 2, 3)$ are the flavor indices):

$$U_{fg} = e^{iP_2/3} [\cos P_1 \delta_{fg} + (e^{-iP_2} - \cos P_1) n_f n_g + \sin P_1 \epsilon_{fgh} n_h], \quad (59)$$

where $P_{1,2}(r)$ are spherically-symmetric profile functions. This ansatz is used to describe strangeness -2 dibaryons.⁴⁵ We shall not consider it here but concentrate on the usual baryons for which the hedgehog ansatz (57) or (58) is appropriate.

Let us first discuss restrictions on the best profile function $P(r)$ which should minimize the nucleon mass (56). What is the asymptotics of $P(r)$ at large r ? To answer this question one has to know the behavior of $E_{\text{field}}[\pi]$ for slowly varying pion fields. Using Eq. (55) as a starting point one can work out the derivative expansion of the functional $E_{\text{field}}[\pi]$ similar to that for the full $E_{\chi\text{L}}$, see subsection 3.1. We have^{37,24}

$$E_{\text{field}}[\pi] = \frac{F_\pi^2}{4} \int d^3\mathbf{x} \text{Tr} L_i L_i - \frac{N_c}{192\pi^2} \int d^3\mathbf{x} [2 \text{Tr} (\partial_i L_i)^2 + \text{Tr} L_i L_j L_i L_j] \\ + \text{higher derivative terms}, \quad L_i = iU \partial_i U^\dagger. \quad (60)$$

Substituting here the hedgehog ansatz one gets a functional of the profile function $P(r)$; varying it one finds the Euler–Lagrange equation valid for slowly varying profiles, in particular, for the tail of $P(r)$ at large r . It follows from this equation that $P(r) = A/r^2$ at large r . The second contribution to the nucleon mass, $E_{\text{level}}[\pi]$, does not alter this derivation since the bound-state wave function has an exponential but not power behavior at large r . Actually we get the pion tail inside the nucleon, and the constant A is related to the nucleon axial constant. This relation is identical to the one found in the Skyrme model:¹¹

$$g_A = \frac{8\pi}{3} A F_\pi^2. \quad (61)$$

The exponentially decreasing wave function of the bound-state level does not change this equation, as well as the Goldberger–Treiman relation for the pion-nucleon coupling constant,

$$g_{\pi NN} = \frac{g_A \mathcal{M}_N}{F_\pi} = \frac{8\pi A}{3} \frac{\mathcal{M}_N}{F_\pi}. \quad (62)$$

Furthermore, it follows from the next four-derivative term in Eq. (60) that the $1/r^4$ correction to $P(r)$ at large r is absent!^{37,24} It means probably that the pion tail inside the nucleon is unperturbed to rather short distances.

The second important question is what should we choose for $P(0)$? As explained in subsection 3.2, a quantity which guarantees a deeply-bound state in the background pion field is the winding number of the field, Eq. (42). Substituting the hedgehog ansatz into Eq. (42) one gets

$$N_{\text{wind}} = -\frac{2}{\pi} \int_0^\infty dr \sin^2 P(r) \frac{dP(r)}{dr} = -\frac{1}{\pi} \left[P(r) - \frac{\sin 2P(r)}{2} \right]_0^\infty. \quad (63)$$

Since $P(\infty) = 0$ the way to make this quantity unity is to choose $P(0) = \pi = 3.141592\dots$

An example of a one-parameter variational function satisfying the above requirements is^{37,24}

$$P(r) = 2 \arctan \left(\frac{r_0}{r} \right)^2, \quad A = 2r_0^2. \quad (64)$$

The Dirac hamiltonian (46) in the hedgehog pion field (57) commutes neither with the isospin operator T nor with the total angular momentum $J = L + S$ but only with their sum $K = T + J$ called the ‘grand spin’. The eigenvalue Dirac equations for given value of K^2, K_3 have been derived

in Ref.(24). Generally speaking, there appears a bound-state level with the $K^P = 0^+$ quantum numbers whose energy can be found from solving the Dirac equations for two spherically-symmetric functions j, h ,

$$\begin{aligned}\frac{dh}{dr} &= -M \sin P h + (E_{\text{level}} + M \cos P) j, \\ \frac{dj}{dr} + \frac{2}{r} j &= M \sin P j + (-E_{\text{level}} + M \cos P) h,\end{aligned}\quad (65)$$

with the boundary conditions $h(0) = 1$, $j(0) = Cr$, $h(\infty) = j(\infty) = 0$.

These equations determine one of the two contributions to the nucleon mass, E_{level} . The second contribution, namely that of the aggregate energy of the lower Dirac continuum in the trial pion field, which we have called E_{field} , can be found in several different ways. One way is to find the phase shifts in the lower continuum, arising from solving the Dirac equation for definite grand spin K .^{24,46} Another method is to diagonalize the Dirac hamiltonian (46) in the so-called Kahana–Ripka basis⁴¹ written for a finite-volume spherical box. Both methods are, numerically, rather involved. There exists a third (approximate) method^{37,24} allowing one to make an estimate of E_{field} in a few minutes on a PC. It is based on the interpolation formula for the $E_{\chi\text{L}}$, see Eq. (37).

Let us discuss the qualitative behavior of E_{level} and E_{field} with the soliton scale parameter r_0 assuming for definiteness that the profile is given by Eq. (64).

The trial pion field plays the role of the (relativistic) potential well for massive quarks. The ‘depth’ of this potential well is fixed by the condition $P(0) = \pi$ and is always finite: this is related to the fact that the pion field has the meaning of angles. The spatial size of the trial pion field r_0 plays the role of the ‘width’ of the potential well. It is well known that in three dimensions the condition for the appearance of a bound state is $MVr_0^2 > \text{const}$ where V is the depth of the well and ‘const’ is a numerical constant of the order of unity depending on its concrete shape. In our case $V \approx M$, so the condition that the bound state appears is $r_0M = O(1)$. Therefore, at small sizes r_0 there is no bound state for the Dirac hamiltonian (46), so that E_{level} coincides with the border of the upper Dirac continuum, $E_{\text{level}} = +M$. At certain critical value of r_0 a weakly bound state emerges from the upper continuum. [For the concrete ansatz (64) the threshold value is $r_0M \simeq 0.5$.] As one increases r_0 the bound state goes deeper and E_{level} monotonously decreases. At very large spatial sizes, $r_0 \rightarrow \infty$, E_{level} approaches the lower continuum, its difference from $-M$ falling as $1/r_0^2$.²⁴ The behavior of E_{level} as function of r_0 is plotted in Fig. 4.

The monotonous decrease of E_{level} with the increase of r_0 is a prerogative

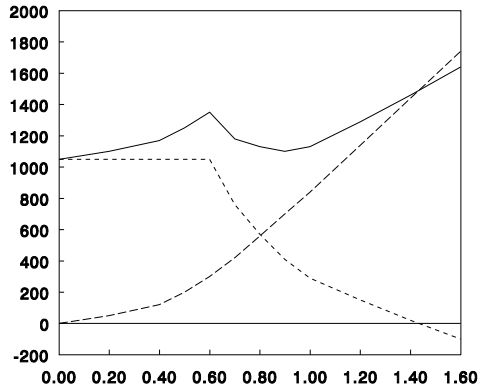


Figure 4: Nucleon mass and its constituents as function of the soliton size. The short-dash line shows $3E_{\text{level}}$, the long-dash line is E_{field} , the solid line is their sum, \mathcal{M}_N .

of the trial pion field with winding number 1. Had it been zero, E_{level} would first go down and then start to go up, asymptotically joining back the *upper* continuum. In the case of $N_{\text{wind}} = -1$ the bound state would travel in the opposite direction: from the lower towards the upper continuum. At $N_{\text{wind}} = n$ as much as n levels would emerge, one by one, from the upper continuum and travel all the way through the mass gap towards the lower one. For the trial pion field of the hedgehog form all these things happen exclusively for states with grand spin $K = 0$.²⁴

Turning now to E_{field} we first notice that for large spatial sizes of the trial pion field one can use the first term in the derivative expansion for E_{field} , see Eq. (60). On dimension grounds one immediately concludes that $E_{\text{field}} \sim F_\pi^2 r_0$ for large r_0 , *i.e.* is infinitely linearly rising in r_0 . At small r_0 a slightly more complicated analysis²⁴ shows that $E_{\text{field}} \sim r_0^3$. On the whole, E_{field} is a monotonously rising function of r_0 shown in Fig. 4.

The nucleon mass, $\mathcal{M}_N = 3E_{\text{level}} + E_{\text{field}}$ (for $N_c = 3$) is also plotted in Fig. 4 taken from Refs.(37,24). One observes a non-trivial minimum for \mathcal{M}_N corresponding to $r_0 \simeq 0.98/M \simeq 0.57$ fm. This is, phenomenologically, a very reasonable value, since from Eqs.(61, 62) one immediately gets $g_A \simeq 1.15$ versus 1.25 (exp.) and $g_{\pi NN} \simeq 13.6$ versus 13.5 (exp.). The nucleon mass appears to be $\mathcal{M}_N \simeq 1100$ MeV with $3E_{\text{level}} \simeq 370$ MeV, $E_{\text{field}} \simeq 730$ MeV. Note that the ‘valence’ quarks (sitting on the bound-state level) come out to be very strongly bound: their wave function falls off as $\exp(-r/0.6 \text{ fm})$, and about 2/3 of the quark mass $M \simeq 350$ MeV is eaten up by interactions with the classical pion field. Relativistic effects are thus essential.

Though the nucleon bound state appears to be somewhat higher than the free-quark threshold, $3M \simeq 1050\text{MeV}$, there are several known corrections to it which are negative. The largest correction to the nucleon mass is due to taking into account explicitly the one-gluon exchange between both ‘valence’ and ‘sea’ quarks; this correction is $O(N_c)$ as is the nucleon mass itself. Numerically, it turns out to be about -200 MeV ⁴⁷ and seems to move the nucleon mass just into the right place.^e

We see thus that the ‘valence’ quarks in the nucleon get bound by a self-consistent pion field whose energy is given by the aggregate energy of the negative Dirac continuum distorted by the presence of the external field. This picture of the nucleon interpolates between the old non-relativistic quark models (which would correspond to a shallow bound-state level and an undistorted negative continuum) and the Skyrme model (which would correspond to a spatially very large pion soliton so that the bound-state level would get close to the lower continuum and the field energy E_{field} would be given just by a couple of terms in its derivative expansion). The reality is somewhere in between: the bound-state level is a deep one but not as deep as to say that all the physics is in the lower Dirac continuum.

Ideologically, this picture of the nucleon at large N_c is somewhat similar to the Thomas–Fermi picture of the atom at large Z . In that case quantum fluctuations of the self-consistent electrostatic field binding the electrons are suppressed by large Z , however corrections go as powers of $Z^{-2/3}$, as contrasted to the N_c^{-1} corrections of the Chiral Quark-Soliton Model. Therefore, the latter model is in a slightly better position in respect to quantum corrections than the former.

Apart from using the large N_c approximation (which is in fact just a technical device needed to justify the use of the classical pion field) the Chiral Quark-Soliton Model makes use of the small algebraic parameter $(M\bar{\rho})^2 \approx 1/3$ where $1/\bar{\rho}$ is the UV cutoff of the E χ L (22). [In the instanton derivation $\bar{\rho}$ is the average size of instantons in the vacuum.] This $\bar{\rho}$ is, roughly, the size of the constituent quark, while the size of the nucleon is, parametrically, $1/M$. The fact that the constituent quark picture works so well in the whole hadron physics finds its explanation in this small numerical parameter. [In the instanton picture it is due to the relative diluteness of the instanton vacuum, which in its turn is related to the ‘accidentally’ large number (11/3) in the asymptotic freedom law.^{14,18}] The small parameter $(M\bar{\rho})^2 \ll 1$ makes it possible to use only quarks with dynamically generated mass and chiral fields as the only

^eThere exist also numerous *quantum* corrections to the nucleon mass of different origin, which are of the order of $O(N_c^0)$. Unfortunately, it is difficult today to treat them in a systematic fashion.

essential degrees of freedom in the range of momenta $k \sim M \ll 1/\bar{\rho}$, and that is exactly the range of interest in the nucleon binding problem.

The above numerics have been obtained from the interpolation formula for E_{field} .^{37,24} Exact calculations of E_{field} performed in⁴⁶ as well as taking more involved profiles with three variational parameters did not lead to any significant changes in the numerics.

Following refs.^{37,24} there had been many calculations of the nucleon mass and of the ‘best’ profile using various regularization schemes and parameters of the chiral model, see⁴⁸ for a review. The effective low-momenta theory (22) comes along with an intrinsic ultraviolet cutoff, in the form of a momentum dependence of the constituent quark mass, $M(k)$. In the instanton approach, it can be shown on general grounds that this is a rapidly decreasing function at momenta of the order of the inverse average instanton size, $1/\bar{\rho} \approx 600$ MeV. However, the present ‘state of the art’ does not allow one to determine this function accurately at all values of momenta – to do so, one would need a very detailed understanding of the QCD vacuum. This places certain restrictions on the kinds of quantities which can sensibly be computed using the effective theory. Those are either finite ones, which do not require an UV cutoff at all, or quantities at most logarithmically divergent. Both type of quantities are dominated by momenta much smaller than the UV cutoff, $k \ll 1/\bar{\rho}$, so one can compute them mimicking the fall-off of $M(k)$ by an external UV cutoff $\Lambda \simeq 1/\bar{\rho}$, using some regularization scheme.^f Fortunately, almost all nucleon observables belong to these two classes. The uncertainty related to the details of the ultra-violet regularization leads to a 15-20% numerical uncertainty of the results, and that is the expected accuracy of the model today.

6 Quantum numbers of baryons

The picture of the nucleon outlined in the previous section is “classical”: the quantum fluctuations of the self-consistent pion field binding N_c quarks are totally ignored. Among all possible quantum corrections to the nucleon mass a special role belongs to the zero modes. Fluctuations of the pion field in the direction of the zero modes cannot be considered small, and one has to treat them exactly. Zero modes are always related to continuous symmetries of the problem at hand. In our case there are 3 zero translational modes and a certain number of zero rotational modes. The latter determine the quantum numbers of baryons; it is here that the hedgehog (or whatever) symmetry of the ansatz taken for the self-consistent pion field becomes crucial.

^fRecently the self-consistent field binding the nucleon has been found using directly the non-local coupling (21).⁴⁹

A general statement is that if the chiral field $U_{\text{cl}}(\mathbf{x})$ minimizes the nucleon mass functional (56), a field corresponding to rotated spatial axes, $x_i \rightarrow O_{ij}x_j$, or to a unitary-rotated matrix in flavor space, $U_{\text{cl}} \rightarrow RU_{\text{cl}}R^\dagger$, has obviously the same classical mass. This is because the functional (56) to be minimized is isotropic both in flavor and ordinary spaces.

Specifically for the hedgehog ansatz [see Eq. (57) for the flavor $SU(2)$ and Eq. (58) for the $SU(3)$] any spatial rotation is equivalent to a flavor rotation. We shall first show it for a more general case of $SU(3)$.

6.1 Case of three flavors

The space-rotating 3×3 matrix O_{ij} can be written as

$$O_{ij} = \frac{1}{2} \text{Tr}(S\tau_i S^\dagger \tau_j) \quad (66)$$

where S is an $SU(2)$ 2×2 matrix and τ_i are the three Pauli matrices. One can immediately check that O_{ij} are real orthogonal 3-parameter matrices with $O_{ij}O_{kj} = \delta_{ik}$ and $O_{ij}O_{ik} = \delta_{jk}$, as it should be.

When one rotates the space putting $n'_i = n_j O_{ji}$ the 2×2 matrix standing in the left upper corner of the ansatz (58) can be written as

$$\begin{aligned} \exp[i(\mathbf{n}' \cdot \boldsymbol{\tau})P(r)] &= \cos P(r) + i(\mathbf{n}' \cdot \boldsymbol{\tau}) \sin P(r) \\ &= S [\cos P(r) + i(\mathbf{n} \cdot \boldsymbol{\tau}) \sin P(r)] S^\dagger. \end{aligned} \quad (67)$$

Therefore, if one considers the hedgehog ansatz (58) rotated *both* in flavor and usual spaces, the latter can be completely absorbed into the former one:

$$R U_{\text{cl}}(\mathbf{O}\mathbf{x}) R^\dagger = \tilde{R} U_{\text{cl}}(\mathbf{x}) \tilde{R}^\dagger \quad (68)$$

with

$$\tilde{R} = R \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}. \quad (69)$$

For that reason it is sufficient to consider rotations only in the flavor space. Hence there are 3 zero rotational modes in the $SU(2)$ and 8-1=7 in the $SU(3)$ flavor case. The rotation of the form $R = \exp(i\alpha\lambda^8)$ commutes with the left-upper-corner ansatz and therefore does not correspond to any zero mode. This will have important consequences in getting the correct spectrum of hyperons.

The general strategy is to consider a slowly rotating ansatz

$$\tilde{U}(\mathbf{x}, t) = R(t) U_{\text{cl}}(\mathbf{x}) R^\dagger(t) \quad (70)$$

and to expand the energy of the bound-state level and of the negative Dirac continuum in ‘right’ (Ω_A) and ‘left’ ($\tilde{\Omega}_A$) angular velocities

$$\Omega_A = -i\text{Tr}(R^\dagger \dot{R} \lambda^A), \quad \tilde{\Omega}_A = -i\text{Tr}(\dot{R} R^\dagger \lambda^A), \quad \Omega^2 = \tilde{\Omega}^2 = 2\text{Tr} \dot{R}^\dagger \dot{R}. \quad (71)$$

Taking into account only the lowest powers in the time derivatives of the rotation matrix $R(t)$ one gets^{24,50} the following form of the rotation lagrangian:

$$L^{\text{rot}} = \frac{1}{2} I_{AB} \Omega_A \Omega_B - \frac{N_c}{2\sqrt{3}} \Omega_8. \quad (72)$$

Here I_{AB} is the $SU(3)$ tensor of the moments of inertia,

$$I_{AB} = \frac{N_c}{4} \int \frac{d\omega}{2\pi} \text{Tr} \left(\frac{1}{\omega + iH} \lambda^A \frac{1}{\omega + iH} \lambda^B \right) \quad (73)$$

where the ω integration contour should be drawn *above* the bound-state energy E_{level} to incorporate the ‘valence’ quarks.

The appearance of a linear term in Ω_8 is an important consequence of the presence of an extra bound-state level emerging from the upper Dirac continuum, which fixes the baryon charge to be unity. In the Skyrme model this linear term arises from the Wess-Zumino term.⁵¹ For simplicity we have written unregularized moments of inertia, though Eq. (73) should be regularized in some way, see *e.g.*⁵⁰

Owing to the left-upper-corner ansatz for the static soliton the tensor I_{AB} is diagonal and depends on two moments of inertia, $I_{1,2}$:

$$I_{AB} = \begin{cases} I_1 \delta_{AB}, & A, B = 1, 2, 3, \\ I_2 \delta_{AB}, & A, B = 4, 5, 6, 7, \\ 0, & A, B = 8. \end{cases} \quad (74)$$

Therefore, the rotational lagrangian (72) can be rewritten as

$$L^{\text{rot}} = \frac{I_1}{2} \sum_{A=1}^3 \Omega_A^2 + \frac{I_2}{2} \sum_{A=4}^7 \Omega_A^2 - \frac{N_c}{2\sqrt{3}} \Omega_8. \quad (75)$$

To quantize this rotational Lagrangian one can use the canonical quantization procedure, same as in the Skyrme model.^{51,29,52,53,54} Introducing eight angular momenta canonically conjugate to ‘right’ angular velocities Ω_A ,

$$J_A = \frac{\partial L^{\text{rot}}}{\partial \Omega_A}, \quad (76)$$

and writing the hamiltonian as

$$H^{\text{rot}} = \Omega_A J_A - L^{\text{rot}} \quad (77)$$

one gets

$$H^{\text{rot}} = \frac{1}{2I_1} \sum_{A=1}^3 J_A^2 + \frac{1}{2I_2} \sum_{A=4}^7 J_A^2 \quad (78)$$

with the additional quantization prescription following from Eq. (76),

$$J_8 = -\frac{N_c}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}. \quad (79)$$

In the Skyrme model this quantization rule follows from the Wess-Zumino term. In our approach it arises from filling in the bound-state level, i.e. from the ‘valence’ quarks. It is known to lead to the selection rule: not all possible spin and $SU(3)$ multiplets are allowed as rotational excitations of the $SU(2)$ hedgehog. Eq. (79) means that only those $SU(3)$ multiplets are allowed which contain particles with hypercharge $Y = 1$; if the number of particles with $Y = 1$ is denoted as $2J + 1$, the spin of the allowed $SU(3)$ multiplet is equal to J .

Therefore, the lowest allowed $SU(3)$ multiplets are:

- octet with spin 1/2 (since there are *two* baryons in the octet with $Y = 1$, the N)
- decuplet with spin 3/2 (since there are *four* baryons in the decuplet with $Y = 1$, the Δ)
- anti-decuplet with spin 1/2 (since there are *two* baryons in the anti-decuplet with $Y = 1$, the N^*)

The next are 27-plets with spin 1/2 and 3/2 but we do not consider them here.

We see that the lowest two rotational excitations are exactly the lowest baryon multiplets existing in reality. The third predicted multiplet, the anti-decuplet, contains exotic baryons which cannot be made of three quarks, most notably an exotic Z^+ baryon having spin 1/2, isospin 0 and strangeness +1. A detailed study of the anti-decuplet performed recently in⁵⁵ predicts that such a baryon can have a mass as low as 1530 MeV and be very narrow. Several experimental searches of this exotic baryon are now under way.

It is easy to derive the splittings between the centers of the multiplets listed above. For the representation (p, q) of the $SU(3)$ group one has

$$\sum_{A=1}^8 J_A^2 = \frac{1}{3}[p^2 + q^2 + pq + 3(p + q)], \quad (80)$$

therefore the eigenvalues of the rotational hamiltonian (78)) are

$$E_{(p,q)}^{\text{rot}} = \frac{1}{6I_2}[p^2 + q^2 + pq + 3(p + q)] + \left(\frac{1}{2I_1} - \frac{1}{2I_2}\right)J(J + 1) - \frac{3}{8I_2}. \quad (81)$$

We have the following three lowest rotational excitations:

$$(p, q) = (1, 1), \quad J = 1/2 : \quad \text{octet, spin } 1/2, \quad (82)$$

$$(p, q) = (3, 0), \quad J = 3/2 : \quad \text{decuplet, spin } 3/2, \quad (83)$$

$$(p, q) = (0, 3), \quad J = 1/2 : \quad \text{anti-decuplet, spin } 1/2. \quad (84)$$

The splittings between the centers of these multiplets are determined by the moments of inertia $I_{1,2}$:

$$\Delta_{10-8} = E_{(3,0)}^{\text{rot}} - E_{(1,1)}^{\text{rot}} = \frac{3}{2I_1}, \quad (85)$$

$$\Delta_{\overline{10}-8} = E_{(0,3)}^{\text{rot}} - E_{(1,1)}^{\text{rot}} = \frac{3}{2I_2}. \quad (86)$$

The appropriate rotational wave functions describing members of these multiplets are given by Wigner finite-rotation functions $D^{8,10,\overline{10}}(R)$.^{50,55}

When dealing with the flavor $SU(3)$ case neglecting the strange quark mass m_s is an over-simplification. In fact, it is easy to incorporate $m_s \neq 0$ in the first order. As a result one gets very reasonable splittings inside the $SU(3)$ multiplets, as well as mass corrections to different observables.^{50,48,55}

In general, the idea that all light baryons are rotational excitations of one object, the ‘classical’ nucleon, leads to numerous relations between properties of the members of octet and decuplet, which follow purely from symmetry considerations and which are all satisfied up to a few percent in nature. The $SU(3)$ symmetry by itself says nothing about the relation between different multiplets. Probably the most spectacular is the Guadagnini formula⁵¹ which relates splittings inside the decuplet with those in the octet,

$$8(m_{\Xi^*} + m_N) + 3m_{\Sigma} = 11m_{\Lambda} + 8m_{\Sigma^*}, \quad (87)$$

which is satisfied with better than one-percent accuracy!

6.2 Case of two flavors

If one is interested in baryons predominantly ‘made of’ u, d quarks, the flavor group is $SU(2)$ and the quantization of rotations is more simple.

In this case the rotational lagrangian is just

$$L^{\text{rot}} = \frac{I_1}{2} \sum_{i=1}^3 \Omega_i^2 = \frac{I_1}{2} \sum_{A=1}^3 \tilde{\Omega}_A^2, \quad (88)$$

where the ‘right’ (Ω_i) and ‘left’ ($\tilde{\Omega}_A$) angular velocities are defined by Eq. (71). This is the lagrangian for the spherical top: the two sets of angular velocities have the meaning of those in the ‘lab frame’ and ‘body fixed frame’. The quantization of the spherical top is well known from quantum mechanics. One has to introduce two sets of angular momenta, S_i (canonically conjugate to Ω_i) and T_A (conjugate to $\tilde{\Omega}_A$). Both sets of operators act on the coordinates of the spherical top, say, the Euler angles. It will be more convenient for us to say that the coordinates of the spherical top are just the entries of the unitary matrix R defining its finite-angle rotation.²⁴

The angular momenta operators S_i, T_A act on R as generators of right (left) multiplication,

$$e^{i(\alpha S)} R e^{-i(\alpha S)} = R e^{i(\alpha \sigma)}, \quad (89)$$

$$e^{i(\alpha T)} R e^{-i(\alpha T)} = e^{-i(\alpha \tau)} R, \quad (90)$$

and satisfy the commutation relations

$$\begin{aligned} [T_A, T_B] &= i\epsilon_{ABC} T_C, & [S_i, S_j] &= i\epsilon_{ijk} S_k, \\ [T_A, S_i] &= 0, & (T_A)^2 &= (S_i)^2. \end{aligned} \quad (91)$$

A realization of these operators is

$$\begin{aligned} S_i &= R_{pk} \left(\frac{\sigma_i}{2} \right)_{kq} \frac{\partial}{\partial R_{pq}}, \\ T_A &= - \left(\frac{\tau_A}{2} \right)_{pk} R_{kq} \frac{\partial}{\partial R_{pq}}. \end{aligned} \quad (92)$$

The rotational hamiltonian is

$$H^{\text{rot}} = \Omega_i S_i - L^{\text{rot}} = \tilde{\Omega}_A T_A - L^{\text{rot}} = \frac{S_i^2}{2I_1} = \frac{T_A^2}{2I_1}. \quad (93)$$

Comparing the definition of the generators (89,90) with the ansatz (70) we see that T_A is the flavor (here: isospin) operator and S_i is the spin operator, since the former acts to the left from R and the latter acts to the right.

The normalized eigenfunctions of the mutually commuting operators S_3 , T_3 and $S^2 = T^2$ with eigenvalues S_3 , T_3 and $S(S+1) = T(T+1)$ are²⁴

$$\Psi_{T_3 S_3}^{(S=T)}(R) = \sqrt{2S+1}(-1)^{T+T_3} D_{-T_3 S_3}^{(S=T)}(R) \quad (94)$$

where $D(R)$ are Wigner finite-rotation matrices. For example, in the $S = T = 1/2$ representation $D_{pq}^{1/2}(R) = R_{pq}$, *i.e.* coincides with the unitary matrix R itself.

The rotational energy is thus

$$E^{\text{rot}} = \frac{S(S+1)}{2I_1} = \frac{T(T+1)}{2I_1} \quad (95)$$

and is $(2S+1)^2 = (2T+1)^2$ -fold degenerate. The wave functions (94) describe at $S = T = 1/2$ four nucleon states (proton, neutron, spin up, spin down) and at $S = T = 3/2$ the sixteen Δ -resonance states, the splitting between them being

$$m_\Delta - m_N = \frac{3}{2I_1} = O(N_c^{-1}) \quad (96)$$

(coinciding in fact with the splitting between the centers of decuplet and octet in the more general $SU(3)$ case, see Eq. (85)).

It is remarkable that the nucleon and its lowest excitation, the Δ , fits into this spin-equal-isospin scheme, following from the quantization of the hedgehog rotation. Moreover, since N and Δ are, in this approach, just different rotational states of the same object, the ‘classical nucleon’, there are certain relations between their properties. These relations are identical to those found first in the Skyrme model¹¹ since they follow from symmetry considerations only and do not depend on concrete dynamics which is of course different in the naive Skyrme model. For example, one gets for the dynamics-independent ratio of magnetic moments and pion couplings¹¹

$$\begin{aligned} \frac{\mu_{\Delta N}}{\mu_p - \mu_n} &= \frac{1}{\sqrt{2}} \simeq 0.71 \quad \text{vs.} \quad 0.70 \pm 0.01 \quad (\text{exp.}), \\ \frac{g_{\pi N \Delta}}{g_{\pi N N}} &= \frac{3}{2} = 1.5 \quad \text{vs.} \quad 1.5 \pm 0.12 \quad (\text{exp.}). \end{aligned} \quad (97)$$

We should mention that there might be interesting implications of the ‘baryons as rotating solitons’ idea to nuclear physics. The low-energy interactions between nucleons can be viewed as interactions between spherical tops

depending on their relative orientation $R_1 R_2^\dagger$ in the spin-isospin spaces.^{56,57} It leads to an elegant description of NN and $N\Delta$ interactions in a unified fashion, and it would be very interesting to check its experimental consequences (as far as we know this has not been done yet). A nuclear medium is then a medium of interacting quantum spherical tops with extremely anisotropic interactions depending on relative orientations of the tops both in the spin-isospin and in ordinary spaces.

This unconventional point of view is supported by the observation⁵⁶ that one can get the correct value of the so-called symmetry energy of the nucleus, $(25 \text{ MeV}) \cdot (N - Z)^2/A$, the numerical coefficient appearing as $1/8I_1 \simeq 25 \text{ MeV}$ where I_1 is the $SU(2)$ moment of inertia; from the $\Delta - N$ splitting (96) one finds $I_1 \simeq (200 \text{ MeV})^{-1}$. We do not know whether the language of spherical tops is fruitful to describe ordinary nuclear matter (probably it is but nobody tried), however it is certainly useful to address new questions, for example whether nuclear matter at high densities can be in a strongly correlated antiferromagnet-type phase.⁵⁷

Finally, let us ask what the next rotational excitations could be? If one restricts oneself to only two flavors, the next state should be a $(5/2, 5/2)$ resonance; in the three-flavor case the next rotational excitation is the anti-decuplet with spin $1/2$, see subsection 6.1. Why do not we have any clear signal of the exotic $(5/2, 5/2)$ resonance? The reason is that the angular momentum $J = 5/2$ is numerically comparable to $N_c = 3$. Rotations with $J \approx N_c$ cannot be considered as slow: the centrifugal forces deform considerably the spherically-symmetrical profile of the soliton field;^{42,58} simultaneously at $J \approx N_c$ the radiation of pions by the rotating body makes the total width of the state comparable to its mass.^{42,59} In order to survive strong pion radiation the rotating chiral solitons with $J \geq N_c$ have to stretch into cigar-like objects; such states lie on linear Regge trajectories with the slope⁴² $\alpha' \approx 1/8\pi^2 F_\pi^2 \approx 1.45 \text{ GeV}^{-2}$. One cannot exclude an intriguing possibility that all large- J hadron states lying on Regge trajectories are actually rotating chiral solitons.⁴²

The situation might be somewhat different in the *three*-flavor case. First, the rotation is, roughly speaking, distributed among more axes in flavor space, hence individual angular velocities are not necessarily as large as when we consider the two-flavor case with $J = 5/2$. Actually, the $SU(2)$ baryons with $J = 5/2$ belong to a very high multiplet from the $SU(3)$ point of view. Second, the radiation by the soliton includes now K and η mesons which are substantially heavier than pions, and hence such radiation is suppressed. Actually, the anti-decuplet seems to have moderate widths⁵⁵ and it is worthwhile to search for the predicted exotic states.

7 Some applications

There exists by now a rather vast literature studying baryon observables in the Chiral Quark-Soliton Model. Baryon formfactors (electric, magnetic and axial), mass splittings, the nucleon sigma term, magnetic moments, weak decay constants, tensor charges and many other characteristics of nucleons and hyperons have been calculated in the model. We address the reader to an extensive review⁴⁸ on these matters.

Here we would like to point out several developments of the Chiral Quark-Soliton Model interesting from the theoretical point of view. The list below is, of course, very subjective.

The study of the spin content of the nucleon in the model has been pioneered by Wakamatsu and Yoshiki.⁶² They showed that the fraction of the nucleon spin carried by the spin of quarks is about 50% (and could be made less): the rest is carried by the interquark orbital moment, the Dirac sea contribution to it being quite essential.

An important question is $1/N_c$ corrections to baryon observables. These can be classified in two groups: one comes from meson loops and is therefore accompanied by an additional small factor $\sim 1/8\pi^2$, the second arises from a more accurate account for the quantization of the zero rotational modes. The corrections of the second type are not accompanied by small loop factors, and may be quite substantial: after all in the real world $N_c = 3$ so a 30% correction is not so small. Such corrections for certain quantities have been fished out in Refs.(63,64) in the two-flavor case and in Ref.(65) for three flavors. These corrections work in a welcome direction: they lower the fraction of nucleon spin carried by quark spins and increase the flavor non-singlet axial constants.

Recent applications of the model are to parton distributions in nucleon, including the so-called skewed distributions, and to the wave function of the nucleon on the light cone. These topics deserve special sections.

8 Nucleon structure functions

The distribution of quarks, antiquarks and gluons, as measured in deep inelastic scattering of leptons, provides us probably with the largest portion of quantitative information about strong interactions. Until recently only the *evolution* of the structure functions from a high value of the momentum transfer Q^2 to even higher values has been successfully compared with the data. This is the field of perturbative QCD, and its success has been, historically, essential in establishing the validity of the QCD itself. However, the initial conditions for this evolution, namely the leading-twist distributions at a relatively low normalization point, belong to the field of non-perturbative QCD. If

we want to understand the vast amount of data on unpolarized and polarized structure functions we have to go into non-perturbative physics.

The Chiral Quark-Soliton Model presents a non-perturbative approach to the nucleons, and it is worthwhile looking into the parton distributions it predicts. Contrary to several models of nucleons on the market today, it is a relativistic field-theoretic model. This circumstance becomes of crucial importance when one deals with parton distributions. It is only with a relativistic field-theoretic model one can preserve general properties of parton distributions such as

- relativistic invariance,
- positivity of parton distributions,
- partonic sum rules which hold in full QCD.

There are two seemingly different ways to define parton distributions. The first, which we would call the Fritsch–Gell-Mann definition, is a nucleon matrix element of quark bilinears with a light-cone separation between the quark ψ and $\bar{\psi}$ operators. According to the second, which we would call the Feynman–Bjorken definition, parton distributions are given by the number of partons carrying a fraction x (the Bjorken variable) of the nucleon momentum in the nucleon infinite-momentum frame. See Feynman’s book⁶⁶ for the discussion of both definitions. In perturbative QCD only the Fritsch–Gell-Mann definition has been exploited as it is very difficult to write down the nucleon wave function in the infinite-momentum frame, which is necessary for the Feynman–Bjorken definition.

Despite the apparent difference in wording, it has been shown for the first time, within the field-theoretic Chiral Quark-Soliton Model, that the two definitions are, in fact, equivalent and lead to identical working formulae for computing parton distributions: in Ref.(67) the first definition has been adopted while in Ref.(68) the second was used. The deep reason for that equivalence is that the main hypothesis of the Feynman–Bjorken parton model is satisfied in the Chiral Quark-Soliton Model, namely that parton transverse momenta do not grow with Q^2 .

Let us point out some key findings of Refs.(67,68).

8.1 *Classification of quark distributions in N_c*

Since the nucleon mass is $O(N_c)$ all parton distributions are actually functions of xN_c . Combining this fact with the known large- N_c behavior of the integrals of the distributions over x one infers that all distributions can be divided into

‘large’ and ‘small’. The ‘large’ distributions are, for example, the unpolarized singlet and polarized isovector distributions, which are of the form

$$D^{\text{large}}(x) \sim N_c^2 f(xN_c), \quad (98)$$

where $f(y)$ is a stable function in the large- N_c limit. On the contrary, the polarized singlet and unpolarized isovector distributions give an example of ‘small’ distributions, having the form

$$D^{\text{small}}(x) \sim N_c f(xN_c). \quad (99)$$

One, indeed, observes in experiment that ‘large’ distributions are substantially larger than the ‘small’ ones.

It should be mentioned that in the large- N_c limit there exist certain strict inequalities between various polarized and unpolarized distributions.⁶⁹

8.2 Antiquark distributions

In the academic limit of a very weak mean pion field in the nucleon the Dirac continuum reduces to the free one (and should be subtracted to zero) while the bound-state level joins the upper Dirac continuum. In such a limit there are no antiquarks, while the distribution of quarks becomes $q(x) = N_c^2 \delta(xN_c - 1)$. In reality there is a non-trivial mean pion field which (i) creates a bound-state level, (ii) distorts the negative-energy Dirac continuum. As the result, the above δ -function is smeared significantly, and a non-zero antiquark distribution appears.

An inevitable consequence of the relativistic invariance is that the bound-state level produces a *negative* contribution to the antiquark distribution.⁹ The antiquark distribution becomes positive only when one includes the contribution of the Dirac continuum. Numerically, the antiquark distribution appears to be sizeable even at a low normalization point, in accordance with phenomenology.

8.3 Sum rules

The general sum rules holding in full QCD are automatically satisfied in the Chiral Quark-Soliton Model: in^{67,68} the validity of the baryon number, isospin,

⁹This is also true for any nucleon model with valence quarks, for example for any variant of bag models. Bag models are essentially non-relativistic, so they fail to resolve this paradox. In order to cure it, one has to take into account contributions to parton distributions from *all* degrees of freedom involved in binding the quarks in the nucleon. That can be consistently done only in a relativistic field-theoretic model, like the one under consideration.

total momentum and Bjorken sum rules has been checked. In fact, it is for the first time that nucleon parton distributions at a low normalization point have been consistently calculated in a relativistic model preserving all general properties.

8.4 *Smallness of the gluon distribution*

As many times stressed in this paper, the whole approach of the Chiral Quark-Soliton Model is based on the smallness of the algebraic parameter $(M\bar{\rho})^2$ where $1/\bar{\rho}$ is the UV cutoff of the low-momenta theory, for example, the average size of instantons in the vacuum. This $\bar{\rho}$ is the size of the constituent quark, while the size of the nucleon is, parametrically, $1/M$. Computing parton distributions in the model one is restricted to momenta $k \sim M \approx 350$ MeV, so that the internal structure of the constituent quarks remains unresolved. There are no gluons in the nucleon at this resolution scale; indeed, the momentum sum rule is satisfied with quarks and antiquarks only.

However, when one moves to the resolution scale of 600 MeV or higher, the constituent quarks cease to be point-like, and that is at this scale that a non-zero gluon distribution emerges. Having a microscopic theory of how quarks get their dynamical masses one can, in principle, compute the non-perturbative gluon distribution in the constituent quarks. What can be said on general grounds is that the fraction of momentum carried by gluons is of the order of $(M\bar{\rho})^2 \approx 1/3$, which seems to be the correct portion of gluons at a low normalization point of about 600 MeV where the normal perturbative evolution sets in.

8.5 *Comparison with phenomenology*

There are several parametrizations of the nucleon parton distributions at a relatively low normalization point, which, after their perturbative evolution to higher momentum transfer Q^2 , fit well the numerous data on deep inelastic scattering. The parametrization most convenient for our purpose is that of Glück, Reya *et al.*^{70,71} who pushed the normalization point for their distributions to as low as 600 MeV, starting from the perturbative side. In Refs.(67,68) parton distributions following from the Chiral Quark-Soliton Model have been compared with those of Refs.(70,71), see Fig. 5. There seems to be a good qualitative agreement though the constituent quark and antiquark distributions appear to be systematically ‘harder’ than those of.^{70,71} This deviation is to be expected since the calculations refer to even lower normalization point where the structure of the constituent quarks themselves is not been taken into account yet, see above.

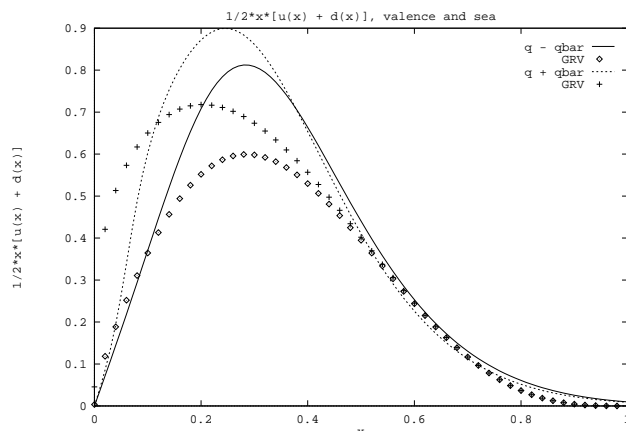


Figure 5: Singlet unpolarized structure functions $x\{[u(x) + d(x)] \pm [\bar{u}(x) + \bar{d}(x)]\}/2$ from Ref.(68) compared to the phenomenological parametrization of Ref.(70). The dashed line corresponds to the sum and the solid line corresponds to the difference of quark and antiquark distributions.

8.6 Flavor asymmetry of the sea

An important feature of the model is the large flavor asymmetry of the sea. Namely, isovector *antiquark* distributions are not small, especially the polarized $\Delta\bar{u} - \Delta\bar{d}$ distribution which is of the leading order in N_c , while the unpolarized $\bar{u} - \bar{d}$ distribution is of the subleading order. Phenomenologically, it is a welcome feature but which is hard to accommodate in other nucleon models. In the Chiral Quark-Soliton Model it arises in a most natural way since the nucleon is bound by the isospin 1 pion field.

In Fig. 6 we present the calculated $\bar{u} - \bar{d}$ distribution⁷² as compared to the experimental data. See⁷³ for the latest discussion of the consequences of large flavor asymmetry of the sea on the description of the data.

Finally, we mention that the Chiral Quark-Soliton Model has been used to make predictions on the transversity spin distributions.⁷⁴

9 Skewed parton distributions

Recently a new type of parton distributions⁷⁶⁻⁸⁰ has attracted considerable interest, the so-called skewed (also called non-forward, off-forward, non-diagonal) parton distributions (the SPD's) which are generalizations of (i) usual parton distributions, (ii) distribution amplitudes and (iii) of the elastic nucleon form

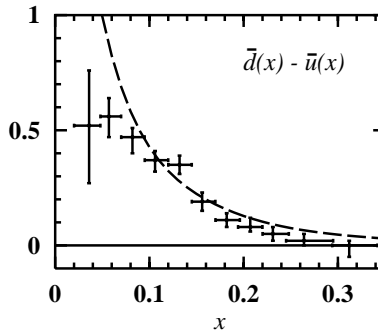


Figure 6: The antiquark asymmetry $\bar{d}(x) - \bar{u}(x)$ in the proton computed in Ref.(72) and perturbatively evolved to $Q = 7.35$ GeV, as compared to the FNAL E866 data of Ref.(75).

factors, in the case of vacuum quantum numbers in the t channel. At the same time there is a wide range of processes where one probes skewed densities which do not have a diagonal analog.⁸⁰

The SPD's are not accessible in standard inclusive measurements. However, they can be measured in diffractive photoproduction of the Z boson,⁷⁶ in diffractive electroproduction of vector mesons,^{77,80,81} in deeply-virtual Compton scattering^{78,79,81} and in hard exclusive electroproduction of mesons at moderate values of Bjorken x .^{80,82}

The number of papers devoted to skewed distributions is rather large by now and continue to grow rapidly. Computation of the SPD's at a low normalization point in a realistic model of nucleons is of great importance. While the usual parton distributions have been measured in a variety of different experiments and can be confronted *a posteriori* with model calculations, in the case of the off-forward distributions the situation is opposite: model calculations of the distributions at a low normalization point are required to determine the very feasibility of measuring the SPD's.

First model calculations of skewed distributions were performed in the bag model.⁸³ However, bag models encounter severe problems in applications to parton distributions. It is usually assumed that the three quarks in the bag give rise only to quark distributions; however, they produce also an antiquark distribution with a *negative* sign, as an inevitable consequence of relativistic invariance. To overcome this problem one needs to add the contribution from forces which bind the quarks – in the case of the bag model it is the mysterious bag surface whose ‘structure function’ will hardly be ever defined. The Chiral

Quark-Soliton Model has no difficulties of that kind, and skewed distributions have been calculated in the model.⁸⁴ They appear to be *qualitatively* different from those of the bag model. We emphasize that this calculation is done in a self-consistent, relativistic-invariant model which ensures the positivity of both quark and antiquark distributions.

Skewed parton distributions are defined through non-diagonal matrix elements of the product of quark fields at a light-cone separation. We shall use the notations of Ji^{78,83} for the two functions depending on three variables:

$$\begin{aligned} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle P' | \bar{\psi}(-\lambda n/2) \psi(\lambda n/2) | P \rangle &= H(x, \xi, \Delta^2) (\bar{u}(P') \not{n} u(P)) \\ &+ \frac{1}{2\mathcal{M}_N} E(x, \xi, \Delta^2) (\bar{u}(P') i\sigma^{\mu\nu} n_\mu \Delta_\nu u(P)). \end{aligned} \quad (100)$$

Here n_μ is a light-cone vector, $n^2 = 0$, $n \cdot (P + P') = 2$, and Δ is the four-momentum transfer, $\Delta = P' - P$. \mathcal{M}_N as usually denotes the nucleon mass, and $u(P)$, $\bar{u}(P')$ are the Dirac spinors of the nucleon in the initial and final states. The off-forward quark distributions, $H(x, \xi, \Delta^2)$ and $E(x, \xi, \Delta^2)$, are regarded as functions of the variable x , the square of the four-momentum transfer Δ^2 and of its longitudinal component $\xi = -(n \cdot \Delta)$.

In the forward case, $P = P'$, both Δ and ξ are zero, and the second term on the r.h.s. of Eq. (100) vanishes. The function H becomes the usual parton distribution function,

$$H(x, \xi = 0, \Delta^2 = 0) = q(x). \quad (101)$$

On the other hand, taking the first moment of Eq. (100) one reduces the operator on the l.h.s. to the local vector current. The dependence of H and E on ξ disappears, and the functions reduce to the usual electric and magnetic form factors of the nucleon,

$$\begin{aligned} \int_{-1}^1 dx H(x, \xi, \Delta^2) &= F_1(\Delta^2), \\ \int_{-1}^1 dx E(x, \xi, \Delta^2) &= F_2(\Delta^2). \end{aligned}$$

The large N_c limit which is the basis of all calculations in the Chiral Quark-Soliton Model allows to calculate off-forward distributions in the region $x, \xi \sim 1/N_c$, $\Delta^2 \sim N_c^0$. Also, it introduces a classification of the skewed distribution in N_c : in the leading order in N_c only isosinglet part of $H(x, \xi, \Delta^2)$ and isovector part of $E(x, \xi, \Delta^2)$ appear. Isovector $H(x, \xi, \Delta^2)$ and isosinglet

$E(x, \xi, \Delta^2)$ are subleading in N_c : they appear after taking into account the finite angular velocity of the soliton rotation.

The calculation of the off-forward parton distributions proceeds in much the same way as that of the usual parton distributions. A typical off-forward distribution obtained from the Chiral Quark-Soliton Model is plotted in Fig. 7. Notice the oscillating character of the distribution with sharp crossovers near the points $x = \pm\xi/2$. It can be shown on general grounds⁸⁴ that they should be present almost in any self-consistent field-theoretic model.^h At small Δ (that is in the forward limit) these points divide the interval $-1 \leq x \leq 1$ into three regions: (i) $x \geq \xi/2$, where the skewed distribution $H(x, \xi, \Delta^2)$ is close to the quark parton distribution, (ii) $x \leq -\xi/2$ where it is close to the antiquark parton distribution (with a minus sign) and (iii) the intermediate region where it has a sharp crossover. The crossover is due to the fact that both quark and antiquark distributions should be positive. It can be shown⁸⁴ that the dependence of the quark mass $M(k)$ on the quark virtuality converts a jump (with infinite derivative) in the skewed distributions into a sharp crossover with the width of the order $\sim (1/M\bar{\rho})^2$.

The fast crossover of $H(x, \xi, \Delta^2)$ at $x = \pm\xi/2$ may have interesting physical implications. For example, it may lead to a considerable enhancement of the deeply-virtual Compton scattering cross section: parametrically it is enhanced by a factor $(\log M\bar{\rho})^2$.

Applications of the Chiral Quark-Soliton Model to *polarized* skewed distributions can be found in Ref.(85). The model predicts a considerable enhancement of the hard exclusive production of charged pions^{86,87} accompanied by a large azimuthal spin asymmetry.⁸⁶ Also, it appears possible to obtain certain model-independent relations between different skewed distributions (based on the large N_c limit), to calculate the transition $N \rightarrow \Delta$ distributions, etc.

10 Light-cone wave function of the nucleon

Probably, the most complete information about processes at high energies involving nucleons can be extracted from its wave function on the light cone or, else, in the infinite momentum frame. Using the infinite momentum frame allows one to separate the wave function of the nucleon from vacuum fluctuations. The wave function is, generally, a Fock vector which determines amplitudes to find inside the nucleon three quarks, or three quarks plus a quark-antiquark pair, or three quarks plus one gluon, etc. The wave function squared gives the probability to find quarks inside the nucleon with given fractions of nucleon longitudinal momentum and some transverse momenta. Integrating it over

^hThey are absent in bag model calculations.⁸³

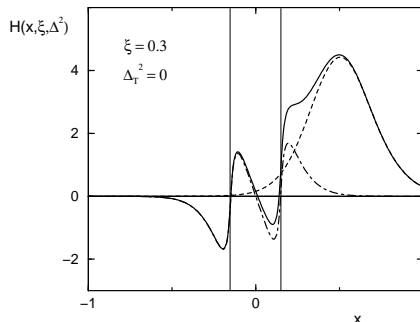


Figure 7: Prediction for the isosinglet skewed distribution $H(x, \xi, \Delta^2)$ for $\Delta_T^2 = 0$ ($\Delta_T^2 \equiv -\Delta^2 - \xi^2 M_N^2$) and $\xi = 0.3$. *Dashed line*: contribution from the discrete level; *dashed-dotted line*: contribution from the Dirac continuum; *solid line*: the total distribution (sum of the two). The vertical lines mark the crossover points $x = \pm \xi/2$.

transverse momenta and summing up over all possible configurations with one quark (or antiquark) fixed one arrives to standard parton distributions.

By the wave function in a ‘narrow sense’ $\phi(z_1, z_2, z_3)$ (else called *distribution amplitude*) people usually understand the amplitude to find exactly three quarks inside the nucleon with given fractions z_1, z_2, z_3 of the nucleon longitudinal momentum. It determines the high-energy asymptotics of exclusive processes such as the asymptotics of electromagnetic formfactors, the decays $J/\psi \rightarrow N\bar{N}$, etc. It is rather well known from experiment (see, e.g. ⁸⁸).

The Chiral Quark-Soliton Model presents a unique possibility to determine the whole Fock state vector of the nucleon on the light cone. The nucleon wave function (as well as parton distributions) are subject to the evolution in perturbation theory, therefore we are actually speaking about the wave function at a low normalization point. This work is in progress, so we present some preliminary results.⁸⁹

In the Chiral Quark-Soliton Model it is easy to define the wave function of the nucleon in the rest frame. Indeed, the nucleon in the model represents quarks in the Hartree approximation in the self-consistent pion field. In this approximation the full wave function is the product of one-particle states,

$$|\Phi_N\rangle = \prod_{\text{occupied}} \int d^3x f^{(n)}(x) \Psi^+(x) |0\rangle, \quad (102)$$

where $f^n(x)$ are the eigenfunctions of the time-independent Dirac operator in

the external pion field, $Hf^{(n)} = E^{(n)}f^{(n)}$, with the hamiltonian (46). The product goes over all occupied levels, i.e. over the discrete ‘valence’ level and over the negative-energy continuum.

Actually we need the wave function not in terms of the Dirac continuum but rather in terms of quarks and antiquarks. It is well known that the wave function (102) corresponds to the so-called coherent exponent:

$$|\Phi_N\rangle = c_{00} \prod_{\text{color}} \int dk F^\lambda(k) \mathbf{a}^{\lambda+}(k_1) \\ \times \exp \int dk_1 dk_2 \mathbf{a}^{\lambda_1+}(k_1) \Theta_{\lambda_1 \lambda_2}(k_1, k_2) \mathbf{b}^{\lambda_2+}(k_2), \quad (103)$$

where $\mathbf{a}^+, \mathbf{b}^+$ are quark and antiquark creation operators. It represents N_c quarks with the wave function $F^\lambda(k)$ and a number of additional quark-antiquark pairs whose wave functions in the external pion field are $\Theta_{\lambda_1 \lambda_2}(k_1, k_2)$; they can be calculated using the Feynman Green function at finite time, which can be constructed from the Dirac equation. As to the single-quark wave function $F^\lambda(k)$ it gets contributions both from the discrete level and from the Dirac continuum described by the pair wave function $\Theta_{\lambda_1 \lambda_2}(k_1, k_2)$.

We next proceed as follows:

- The wave function (103) is translated to the infinite momentum frame. This is possible as the model is relativistically invariant.
- The pair wave functions are calculated using the technique of interpolating formulae described in subsection 4.1. It is interesting to note that higher terms of this expansion give the wave functions with 1, 2, ... additional pions on the light cone.^{90,91} This is to be expected, as pions are the only agents inducing the interaction in the model.
- The result should be averaged with the rotational wave function defined in section 6.2 to project the general wave function onto a given rotational state of the soliton – the nucleon, the Δ , etc.

The three-quark contribution to the (proton, spin up) wave function is:

$$|\Phi_N, \text{proton } \uparrow\rangle = \Phi(z_1, p_{1\perp}, z_2, p_{2\perp}, z_3, p_{3\perp}) |u\uparrow(1) \rangle |u\downarrow(2) \rangle |d\uparrow(3) \rangle. \quad (104)$$

After integrating over transverse momenta of quarks one gets the distribution amplitude. The discrete-level contribution to it has the form:

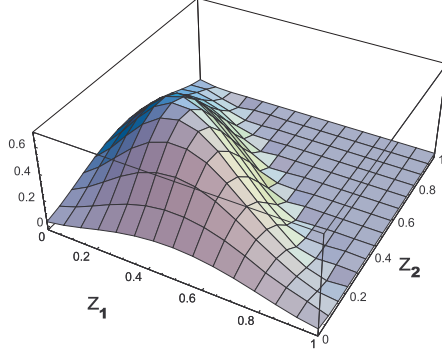


Figure 8: Nucleon wave function $\phi(z_1, z_2, 1 - z_1 - z_2)$. Only the ‘valence’ level contribution is included.

$$\begin{aligned} \phi(z_1, z_2, z_3) = & \int d^2x_{\perp} \{g(z_1, x_{\perp})g(z_2, x_{\perp})g(z_3, x_{\perp}) \\ & + x_{\perp}^2 j(z_2, x_{\perp}) [2g(z_1, x_{\perp})j(z_3, x_{\perp}) - j(z_1, x_{\perp})g(z_3, x_{\perp})]\}, \end{aligned} \quad (105)$$

where

$$g(z, x_{\perp}) = -\frac{\pi^{3/2}}{2} \int dx_3 \exp(-i(z\mathcal{M}_N - E_{\text{level}})) \left[h(r) - \frac{ix_3}{r} j(r) \right], \quad (106)$$

$$j(z, x_{\perp}) = \frac{\pi^{3/2}}{2} \int dx_3 \exp(-i(z\mathcal{M}_N - E_{\text{level}})) \left[\frac{i}{r} j(r) \right]. \quad (107)$$

Here $h(r)$ and $j(r)$ are the wave functions of the discrete level (65) and E_{level} is its energy.

Let us note that the second term in curly brackets in Eq. (105) introduces an asymmetry of the wave function under the exchange of z_i ’s. This effect is due to the finiteness of N_c and appears as a result of averaging over nucleon rotational wave function. At $N_c \rightarrow \infty$ the wave function of the nucleon would be completely symmetric.

The calculated distribution amplitude (105) is presented in Fig. 8. Though not fully symmetric under the interchange of z_i ’s, it is much closer to the asymptotic nucleon wave function,

$$\phi^{\text{asympt}}(z_1, z_2, z_3) = 120 z_1 z_2 z_3 \delta(z_1 + z_2 + z_3 - 1), \quad (108)$$

than the wave function suggested by Chernyak, Ogloblin and Zhitnisky⁹² many years ago on the basis of their analysis of the QCD sum rules.ⁱ It seems to be supported by the data.⁸⁸

11 Conclusions

The Chiral Quark-Soliton Model of baryons is a simple but still a non-trivial reduction of the full-scale QCD at low energies. It emphasizes the spontaneous chiral symmetry breaking in QCD, accompanied by the appearance of the dynamical (or constituent) quark mass. We prefer the word ‘dynamical’: first, because it is, indeed, dynamically generated, second, because it is momentum-dependent.

The momentum dependence of the dynamical quark mass $M(k)$ is the key to understanding why the notion of constituent quarks have worked so remarkably well over 30 years in hadron physics. The point is, the scale Λ at which the function $M(k)$ falls off appears to be much larger than $M(0)$; the former determines the size of constituent quarks while the latter determines the size of hadrons. As a matter of fact these two distinctive scales come neatly from instantons.

The Chiral Quark-Soliton Model fully exploits the existence of the two distinctive scales: it is because of them it makes sense to restrict oneself to just two degrees of freedom in the nucleon problem, namely, to massless or nearly massless (pseudo) Goldstone pions and to the constituent quarks with a momentum-dependent dynamical mass $M(k)$. The scale Λ actually plays the role of the physical ultra-violet cutoff for the low-energy theory; its domain of applicability is thus limited to the range of momenta $k \sim M < \Lambda$. This is precisely the domain of interest for the nucleon binding problem.

A technical tool simplifying considerably the nucleon problem is the use of the large N_c logic. At large N_c the nucleon is heavy, and one can speak of the classical self-consistent pion field binding the N_c valence quarks of the nucleon together. The classical pion field (the soliton) is found from minimizing the energy of the bound-state level plus the aggregate energy of the lower Dirac continuum in a trial pion field. The ‘valence’ quarks sitting on the bound-state level appear to be strongly bound by the classical pion field. The system is relativistic.

By quantizing the slow rotations of the soliton field in flavor and ordinary spaces one gets baryon states which are rotational excitations of the static ‘classical nucleon’. The classification of the rotational excitations depends on

ⁱThe *pion* wave function at a low normalization point as calculated from the instanton vacuum⁹⁰ also appears to be rather close to its asymptotic form.

the symmetry properties of the soliton field, but not on the details of dynamics. Taking the hedgehog ansatz one gets the following lowest baryon multiplets: octet with spin $1/2$, decuplet with spin $3/2$ (these are, indeed, the lowest multiplets observed in nature) and antidecuplet with spin $1/2$. This last multiplet contain baryons with exotic quantum numbers (in the sense that they cannot be composed of only three quarks); some of them are predicted to be relatively light and narrow resonances, and it would be of great interest to search for such states.

By saying that all lightest baryons are nothing but rotational excitations of the same object, the ‘classical nucleon’, one gets many relations between members of baryon multiplets, which are realized with astonishing accuracy in nature. Especially successful are predictions which do not depend on dynamical quantities (like the values of moments of inertia) but follow from symmetry considerations only, and are therefore shared, *e.g.*, by the Skyrme model. Predictions of the Chiral Quark-Soliton Model which do depend on concrete dynamics are, in general, also in good accordance with reality: the typical accuracy for numerous baryon observables computed in the model is about 15-20%, coinciding with the expected theoretical accuracy of the model. To get a better accuracy one needs a better understanding of the underlying QCD vacuum and of the resulting effective low-energy theory.

To our knowledge, the Chiral Quark-Soliton Model is today the only relativistic field-theoretic model of the nucleon, and this advantage of the model becomes crucial when one turns from static characteristics of the nucleon to the numerous parton distributions. It is impossible to get a consistent description of parton distributions satisfying positivity and sum rules restrictions, without having a relativistic theory at hand and without taking into account the complete set of forces which bind quarks together. The leading-twist parton distributions calculated in the Chiral Quark-Soliton Model refer to a very low normalization point where the structure of the constituent quarks is not resolved yet. They seem to be in qualitative agreement with parametrizations of the DIS data at low Q^2 though, not unnaturally, they appear to be more ‘hard’. One of the striking predictions of the model is large flavor asymmetry of antiquark distributions, which is favored by recent experiments. An even larger asymmetry is predicted for polarized antiquark distributions, but it has not been directly measured so far.

The Chiral Quark-Soliton Model has been recently used to calculate some of the off-diagonal or skewed distribution functions which provide even a more detailed information on the nucleon structure than the usual (diagonal) parton distributions. It predicts a drastic oscillating form of the skewed distribution. Preliminary results have been obtained for the nucleon wave function on the

light cone.

An impressive amount of data on parton distributions has been collected in the past years, and its amount will grow with new types of distributions becoming experimentally accessible. We know how to (perturbatively) evolve parton distributions from high to still higher values of Q^2 but we do not understand how to explain the initial conditions for that evolution, since it involves truly nonperturbative physics. This is a real challenge to the theory of strong interactions. At present the only nonperturbative relativistic field-theoretic model of QCD at low energies is the ‘Nucleons as Chiral Solitons’ model. Therefore, we think that it will play a major role in understanding, describing and predicting data in the years to come.

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